Honors Analysis Chapter 2 2003

NOTE: We use

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \textbf{and} \quad y - y_1 = m(x - x_1)$$

to

determine the equation of a straight line in graphing form:

$$y = mx + b$$

				# 1 R/C 55 - 59 • • • • • • • • • • • • • • • #2 State Question and/or draw picture and state answer to oral ex pg 58-59 • • • • • • • • • • • • • • • # 3 Pg 60-61 1 - 39 odd 2 / 14
School Holiday	#4 Pg 60 - 61 2 - 40 even, , 40 - 46 • • • • • • • • • • • • • • • # 5 R/C Pg 63 - 66 • • • • • • • • • • • • • • • #6 Write Q & A for oral ex Pg 67 **Tell them to use** $y - y_1 = m(x - x_1)$ 2 / 18	# 7 Pg 67 - 68 1 - 30 • • • • • • • • • • • • • • • # 8 Pg 62 47 B Be sure you make an "honors" effort on this! **Parallel, Perpendicular lines** 2 / 19	# 9 Pg 78 - 79 3, 6, 12, 13, 15 - 17 **Define Parabola** **Explain pg 62 # 47B** 2 / 20	# 10 Pg 74 23 - 27 • • • • • • • • • • • • • • • # 11 Pg 79 18 - 23 • • • • • • • • • • • • • • • # 12 Pg 68 33 **Lecture on Max/Min Set-up and solve. Do eg pg 72. let x = ... What is to be max-ed or min-ed** **Quiz #1** 2 / 21
# 13 "A Parabolic Trip Down Memory Lane." The set-up *is* most important! **– b/2a styled max/min pblms Quiz #2** 2/ 24	# 14 Pg 104 - 105 1 - 5, 7 **I showed pblms 9, 10** 2/15	#15 Pg 85 13 - 18 Pg 92 1, 9, 10, 11 **"El Quizzo Extraordinaire"** **Lecture: Synth Div, Synth substitution, Factor thm, Give Take Home Quiz** 2 / 26	# 16 Pg 97 1, 2, 5 Pg 104 - 105 8 - 11 **also solve:** $24x^5 - 42x^4 - 77x^3 + 56x^2 + 60x = 0$ **also...** **restate factor thm and Rational Root thm** 2 / 27	**Important Lecture** **including Decartes Rule of signs up/low bounds** Begin # 17 RRTS 1 - 5 2/28
# 18 RRTS 6 - 10 **Have 1 - 5 from RRTS put up. point out upper bounds, etc.** 3 / 3	# 19 RRTS 11 - 15 3 / 4	# 20 RRTS 15 - 20 3 / 5	Put Last years Exam on chalk boards 3 / 6	EXAM CHAPTER 2 2003 3 / 7

A parabolic trip down Memory Lane

Set up properly by defining a variable, drawing a picture where possible, creating a second degree equation which describes the item to be maximized or minimized. Find the required value. State your conclusion using complete sentences.

1. Find two numbers whose sum is 10 and whose product is as great as possible.

2. Find the area of the rectangle of maximum area with perimeter 20 inches.

3. If an object is thrown upward with an initial velocity of 128 feet per second from an height above the ground of 20 feet, its height in feet above the ground after t seconds is given by: $h = 20 + 128t - 16t^2$. What is the maximum height attained by the object? When will it hit the ground?

4. If a stone is tossed upward with an initial velocity of 112 feet per second from a height of 700 feet above the surface of Mars, its height h in feet above the surface t seconds later is given by $h = -5.6t^2 + 112t + 700$. For what value of t will the stone be at its maximum height?

5. In a 110-volt circuit having a resistance of 11 ohms, the power W when a current I is flowing is given by $W = 110I - 11I^2$. Determine the maximum power that can be delivered in this circuit.

6. A real-estate operator estimates that the monthly profit p in thousands of dollars from a building is given by $p = -2s^2 + 88s$. What height would he consider most profitable?

7. A rectangular "Main" field is to be enclosed by a fence and divided into two smaller rectangular fields by another fence. Find the dimensions of the "Main" field of greatest area than can be thus enclosed and partitioned with twelve hundred feet of fencing.

8. A homesteader is offered as much land as he can enclose in a rectangular plot, one side of which is to lie along the straight bank of a river. Find the area of the largest such plot he can obtain if he has twenty-four hundred feet of fencing to use and does not fence the river side of his plot.

9. A fruit grower wishes to ship early in the season when prices are high and spoilage is low. He now has 15 tons on hand and estimates that he can add three tones per week by waiting. He gets three hundred dollars per ton now but estimates that this will reduce by twenty dollars per ton for each week he delays. When should he ship to maximize his revenue?

10. A helicopter shuttle service operating between an airport and the center of a city charges a fare of $10 and carries 300 persons per day. The manager estimates that he will lose 15 passengers for each increase of $1 in the fare. How much should the service charge in order to maximize its revenue?

HONORS ANALYSIS CHAPTER 2 (somewhat) helpful hints for homework

Pg 60 - 61 (12) $m = \frac{-2}{3}$ (2, 0) (14) $F(x) = -3x + 6$ (34) $F(x) = mx + b$ $x = mz + b$ (18) $F(x) = 2x - 3$

(35) $f(x) = mx + b$ $g(x) = px + q$ $f(g(x)) = \ldots$ (36) (1, 5) (2 , 7) (37) (0, 20) (2, 30)

(38) Let $v(t) = mt + b$ (1, 96) (2, 64) $m = -32$ $y - 96 = -32(x - 1)$ ==> $y = -32x + 128$ so initial velocity (ie $t = 0$) = ... $v\left(\frac{7}{2}\right) = ?$ etc. (39) (0, 1600) (25, 800) so $m = 28$?

Pg 62 (40) is (0, b) ==> (b, 0) the only version ? Draw some examples !!

Pg 68 (14) given line has $m = 0$ so $y - 2 = 0(x - 3)$ etc. (24) $\frac{2t - t}{t + 1 - t} \overset{\text{set}}{=} \frac{1}{2}$ etc (28) $\frac{1}{k} \overset{\text{set}}{=} \frac{1}{2}$

(33) coord of P_2 is (x, y) coord of P_2 could also be $(x_0 + x',\ y_0 + y')$

Given equation $Ax + By + C = 0$ where x is some x value along x-axis. Well, $(x_0 + x')$ is a point so, $A(x_0 + x') + B(y_0 + y') + C = 0$ which becomes

$Ax' + By' + (Ax_0 + By_0 + C) = 0$ Note x' is the x change from P_1 to P_2 y' marks a similar change for y. According to pblm 47 pg 62, distance from origin to line is that constant. NOW we have the same type of equation. The corresponding parts are the constant terms

Pblm 47 ==> C Pblm 33 ==> $(Ax_0 + By_0 + C)$

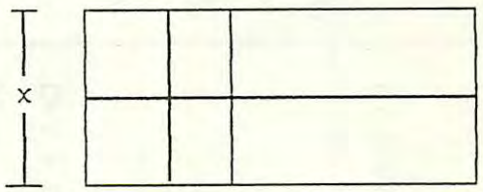
Pg 72 Function = $A\left(x - \left(\frac{-b}{2a}\right)\right)^2 + C - \frac{b^2}{4a}$ Question: What is max area ie a max value of area $C - \frac{b^2}{4a}$

when / if / cause — what / then / effect

Question: When does extremum occur ? $\frac{-b}{2a}$

Pg 74 method: (1) Draw a picture if possible (2) Define variables with "LET" statements (3) write a function which describes the item to be maximized or minimized.

eg: Rectangular field --- short side is x Total fence is 120. Find max area enclosed



(1) Total of vertical fence is 4x
(2) How much is left ? 120 – 4x
(3) EACH horizontal fence one-third of total left ie $\frac{1}{3}(120 - 4x)$
(4) Max area ? Well Don't worry about MAX. YOU cannot control it. What's left ? AREA --- How do you find area ? L X W ?

(5) Define function ==> $A(x) = x\left(\frac{1}{3}(120 - 4x)\right) = 40x - \frac{4}{3}x^2$

(6) What was the question ? Was it what length sides CAUSE maximum area OR was it What is maximum area ?

Pg 79 (25) Let P = distance from vertex to directorix and from vertex to focus.

Def of Parabola The set of points equidistant from a fixed point (called the focus) and a fixed line (called the directorix).

$\overline{P_2 f} = \sqrt{((k - p) - y)^2 + (h - x)^2}$ $\overline{P_1 P_2} = \sqrt{((k + p) - y)^2 + (x - x)^2}$

$\sqrt{((y - k) + p)^2 + (x - h)^2} = \sqrt{((y - k) - p)^2}$ Keep the $(y - k)$ terms together --- they will subtract off.

This all simplifies to $4p(y - k) = -(x - h)^2 \Rightarrow y - k = \frac{-1}{4p}(x - k)^2 \Rightarrow$ etc (the negative means the parabola opens downward.)

(26) Complete the square. $4p = \frac{1}{A}$

ANL Ch2
Quiz #1
on DAY HW fm Pg 60 is collected.

8 minutes.

write an equation in slope-intercept form...

① $m=4$ $(-5,6)$ $\quad y-6=4(x+5)$
$y=4x+26$

② $(1,30)$ $(-1,22)$ $\quad \frac{30-22}{1+1}$
$\frac{8}{2}$
$m=4$
$y-30=4(x-1)$
$y=4x+26$

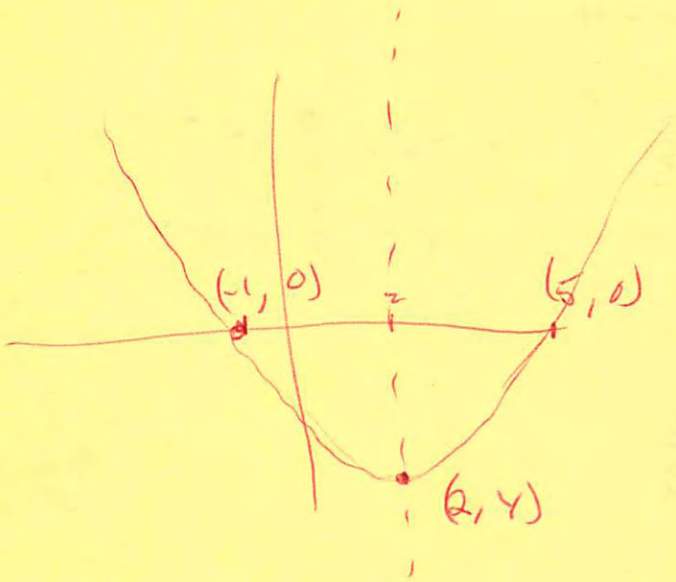
③ $m=\frac{3}{2}$ x intercept $\frac{2}{3}$
$(\frac{2}{3},0)$
$y-0=\frac{3}{2}(x-\frac{2}{3})$
$y=\frac{3}{2}x-1$

④ $m=\frac{3}{2}$ y intercept $\frac{2}{3}$

$y=\frac{3}{2}x+\frac{2}{3}$

⑤ y intercept 5
parallel to $2y+6x=-17$
$y=-3x-\frac{17}{2}$
$m=-3$
$y=-3x+5$

Midpt of x intercepts gives a vertical line on which the vertex must lie.



Given: x intercepts of a parabola are

$(-1, 0)$ and $(5, 0)$

Find the coordinates of the focus and of the vertex of a parabola with those x intercepts.

one could do negative

$0 \overset{\text{set}}{=} (x+1)(x-5)$

$x^2 - 4x - 5$

$x^2 - 4x + 4 - 9$

$y = (x-2)^2 - 9$

OR

$(x+1)(5-x) \overset{\text{set}}{=} 0$

$-x^2 + 4x + 5 = 0$

$-(x^2 - 4x + 4) + 5 + 4 = 0$

$-(x-2)^2 + 9 = 0$

$y = -(x-2)^2 + 9$

Vertex $(2, -9)$ or $(-2, 9)$

$\frac{1}{4c} = 1$, $c = \frac{1}{4}$ or $\frac{1}{4c} = -1$ $c = -\frac{1}{4}$

focus $(2, -\frac{35}{4})$ or $(-2, \frac{35}{4})$

There is an infinite number of parabolas which meet this criteria. Had the vertex also been given would make it unique.

Everything = +

one set ✓

Multiplied $(x+1)(x-5)$ 0

after effort

BLANK =

AnL Ch 2

NO calculators.

@ 4:12 #2 10 min

changed to "Take Home" Div. 2 But given again one step.

Given $y = -\frac{1}{2}x^2 - x + 4$

Compute the equation for each of the 3 lines which pass through the focus of f and through an axis intercept of f.

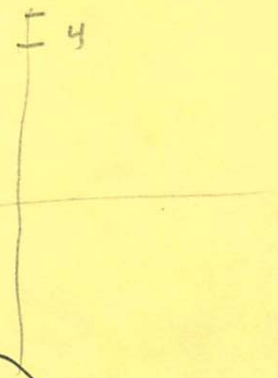
$\left(\frac{-b}{2a}, c - \frac{b^2}{4a}\right)$ $y = -\frac{1}{2}(x^2 + 2x + 1) + 4 + \frac{1}{2}$ (3)

$= -\frac{1}{2}(x+1)^2 + \frac{9}{2}$ $\frac{1}{4c} = \frac{1}{2} \therefore c = \frac{1}{2}$

① $v(-1, \frac{9}{2})$

dir $y = 5$

② f $(-1, 4)$



$f(0) = 4$ (6), (7)

if y intercept

$y = -\frac{1}{2}x^2 - x + 4 > 0$ SW

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

$x = \{-4, 2\}$ (4) (5)

Pens Scoring

all 3 equations

+	10, 11, 12
✓	8, 9
0	6, 7
−	4, 5
=	0, 1, 2, 3

(0, 4)
(-1, 4) } $y = 4$ (12)

(-4, 0)
(-1, 4)

$\frac{4-0}{-1+4} = \frac{4}{3}$ (8)

$y = \frac{4}{3}x + \frac{16}{3}$ (9)

(2, 0)
(-1, 4)

$\frac{4-0}{-1-2} = \frac{-4}{3}$ (10)

$y = -\frac{4}{3}x + \frac{8}{3}$ (11)

Tomar Tu Casa Quizzo

9,10 +
8, 8 ✓
4,5,6, 0
1 2 3 −
0 =

Name _____

Analysis Chapter 2
Mr. Mumaugh

I

Given: $f(x) = -\frac{1}{2}x^2 - x + 4$

a) Use the method of "completing the square" to put $f(x)$ into "graphing form".
b) Compute the coordinates of thethree axis intercepts.
c) Compute the equation for each of the three lines which pass through the focus of $f(x)$ and through an axis intercept.

(7)

a)

$f(x) = -\frac{1}{2}(x^2 + 2x + \quad) + 4$

(1) $= -\frac{1}{2}(x^2+2x+1) + 4 + \frac{1}{2}$

$= -\frac{1}{2}(x+1)^2 + \frac{9}{2}$

c) $= \frac{-1}{4c}(x-h)^2 + k$

$c = \frac{1}{2}$ (3) on focus $(-1, 4)$

b) $f(0) = 4$ (0, 4)

$-\frac{1}{2}x^2 - x + 4 = 0$ SJ (−4, 0) (2, 0)

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

Focus $(-1, 4)$ $(0, 4)$ $y = 4$ (4)

$(-1, 4)$ $(-4, 0)$ $\frac{4-0}{-1+4} = \frac{4}{3}$ $y - 0 = \frac{4}{3}(x+4)$ (5)

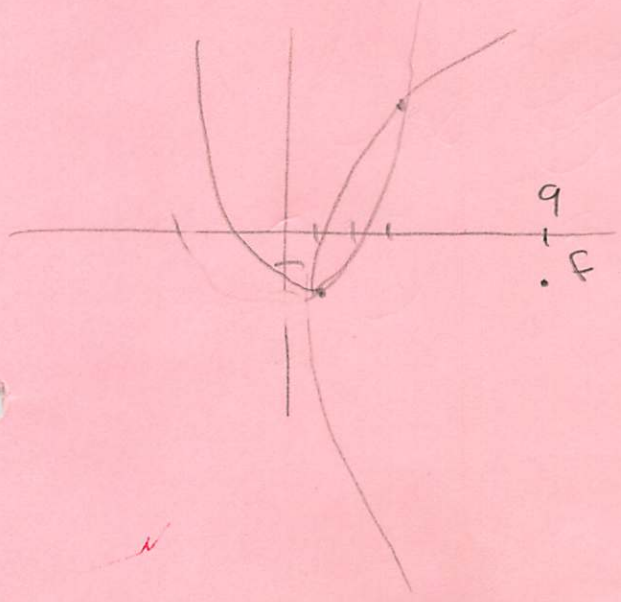
$y = \frac{4}{3}x + \frac{16}{3}$

$(-1, 4)$ $(2, 0)$ $\frac{4-0}{-1-2} = \frac{-4}{3}$ $y - 0 = -\frac{4}{3}(x-2)$ (6)

$y = -\frac{4}{3}x + \frac{8}{3}$

II

Find the coordinates of the focus of the parabola which has vertex at $(1, -2)$ and passes through the point $(3, 6)$.



$y = \frac{1}{4c}(x-h)^2 + k$ $V = (1, -2)$

$6 = \frac{1}{4c}(3-1)^2 - 2$ $f = (1, -\frac{15}{8})$ (7) (8)

$32c = 4$

$c = \frac{1}{8}$

$x = \frac{1}{4c}(y-k)^2 + h$ $V = (1, -2)$

$3 = \frac{1}{4c}(6+2)^2 + 1$ $f = (9, -2)$ (9) (10)

$8c = 8^2$

$c = 8$

Show Pblm 9. ~~Give~~ Leave on Boards. Give 5 min Quiz "do Pblm 10"

(9) let w = weeks to wait

$15+3w$ = New Tons of fruit

$300-20w$ = New price per ton

$$f(x) = \underbrace{(15+3w)}_{\#\text{ of items}}\underbrace{(300-20w)}_{\text{value each}}$$

$$= 4500 + 600w - 60w^2$$

$$\frac{-600}{-2\cdot 60} = 5$$

$$\frac{-b}{2a}$$

when? refers to weeks

The grower should wait 5 weeks.

(10) let x = fare increase

$10+x$ = New fare

$300-15x$ = New # of riders

$$f(x) = \underbrace{(300-15x)}_{\#\text{ of items}}\underbrace{(10+x)}_{\text{value each}}$$

$$= 3000 + 150x - 15x^2$$

$$\text{Rev} = (\#\text{ of items})(\text{value ea})$$

?

change? refers to fare.

$$\frac{-b}{2a}$$

$$\frac{-150}{-2\cdot 15} = 5$$

New fare is $10+5 = \$15$.

Scoring 1 pt ea

3 lets

Generic fctn

fctn

$\frac{-b}{2a}$

Sentences

5 +

4 ✓

3 0

2 −

0,1 =

(2) $m=-1 \quad (0,3)$

$y-3=-x$

$x+y-3=0$

(4) $m=-\frac{1}{2} \quad (1,2)$

$y-2=-\frac{1}{2}(x-1)$

$-2y+4=x-1$

$x+2y-5=0$

(6) $(2,-1), (-3,-4)$

$\frac{-4+1}{-3-2}=\frac{3}{5}$

$y+1=\frac{3}{5}(x-2)$

$5y+5=3x-6$

$3x-5y-11=0$

(8) $m=-2 \quad (\frac{1}{2}, 0)$

$y=-2(x-\frac{1}{2})$

$y=-2x+1$

$2x+y-1=0$

(10)

(2) 3

(4) 7

(6) $\frac{1}{3}$

(8) $-\frac{2}{3}$

(10) 1, -3

(12) 4 0

14 $\frac{1}{2}$ 3

16 2 3

18 1 -4

20 $-\frac{2}{2}$ -2

$y - y - y$

$y - 1 = y$

$3y = -2x - 6$

$y = -\frac{2}{3}x - 2$

ANALY Quiz after pg 60 chpt 2.

Write an equation in slope-intercept form that is suggested by the following information:

① $m = 4$ $(\pm 5, 6)$
② $(1, 30)$ $(-1, 22)$
③ $m = \frac{3}{2}$ Y-intercept $\frac{2}{3}$
④ $m = \frac{3}{2}$ x-intercept $\frac{2}{3}$

} 2 pts ea

⑤ Y-intercept 5, Parallel to: $2Y + 6x = -17$
⑥ Y-intercept -3, Perpendicular to: $(x-2)^2 = 1x^2$
⑦ x-intercept 6 & Perpendicular to $2Y - 6 = x$
⑧ x-intercept 3 & Y-intercept -5.

} 3 at ea.

① $Y - 6 = 4(x + 5)$
$Y = 4x + 26$

② $\frac{30 - 22}{1 + 1} = \frac{8}{2} = 4 = m$
$Y - 30 = 4x - 4$
$Y = 4x + 26$

8 ✓
6,7 ✓
4,5 0
3,2 −
0,1 =

③ $Y = \frac{3}{2}x + \frac{2}{3}$

④ $Y = \frac{3}{2}(x - \frac{2}{3}) \Rightarrow Y = \frac{3}{2}x - 1$

⑤ $2x = 6x - 17$
$m = -3$
$Y = -3x + 5$

⑥ $Y = -3$

⑦ $(6, 0)$ $2Y = x + 6$
$Y = \frac{1}{2}x + 3$
$m = -2$
$Y = -2(x - 6)$
$Y = -2x + 12$

⑧ $(3, 0), (0, -5)$
$\frac{-5}{-3} = \frac{5}{3} = m$
$Y = \frac{5}{3}x \pm 5$

12
6

① $m = 4 \quad (-5, 6)$

$y - 6 = 4(x + 5)$

$y = 4x + 26$

② $(1, 30) \quad (-1, 23)$

$\frac{30 - 23}{1 + 1} = \frac{7}{2}$

$y - 30 = \frac{7}{2}(x - 1)$

$y = \frac{7}{2}x + \frac{53}{2}$

$\frac{7}{2}$ 60

③ $m = \frac{3}{2}$ y int $\frac{3}{2}$

$y = \frac{3}{2}x + \frac{3}{2}$

④ $m = \frac{3}{2}$ x int $\frac{2}{3}$

$y = \frac{3}{2}(x - \frac{2}{3})$

$y = \frac{3}{2}x - 1$

⑤ $(5, 0) \quad m = +\frac{1}{3}$

$2y = -6$

$x = -3$

$y = \frac{1}{3}x + 5$

⑥ $x^2 - 6x + 9 = y^2$

$y = 3$

$y = -3$

⑦ $(6, 0)$ $2y = x$

$y = \frac{1}{2}x$

$m = -2$

$y = -2(x - 6)$

$x = -2x + 12$

⑧ $(-3, 0) \quad (0, -5)$

$\frac{-5 - 0}{0 + 3} = -\frac{5}{3}$

$y = -\frac{5}{3}x - 5$

(12) $2x+3y=0$ $\quad 3x+2y=6$

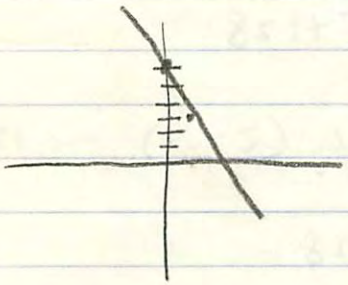
$y=-\frac{2}{3}x$ $\quad$ if $y=0$ $x=2$

$m=-\frac{2}{3}$ $\quad (2,0)$ $\quad y=-\frac{2}{3}(x-2)$

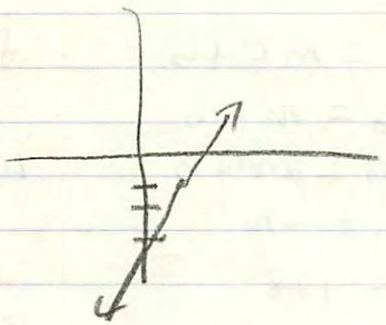
$y=-\frac{2}{3}x+\frac{4}{3}$

(14) $f(x)=3(2-x)$

$=-3x+6$



(18) $f(x)=2|x|-3$



34

$f(x)=mx+b \quad m\neq 0$

$x=mz+b$

$mz=x-b$

$z=\frac{x}{m}-\frac{b}{m}$

35 $f(x)=mx+b$

$g(x)=px+q$

$g\circ f=p(mx+b)+q$

$=mpx+pb+q$

36

$y=mx+b$

$5=m\cdot 1+b$ $\quad 7=2m+b$

$5=m+b$

$2=m$

$b=3$

f(x) $S(x)=2x+3$

no weight $\Rightarrow$ 3 feet

$8=2x+3$

$5=2x$

$\frac{5}{2}=x$

$2\frac{1}{2}$ lbs required.

37 $b = 20$ $30 = 2x + 20$ $f(x) = 5x + 20$

$x = 5$

1 ft → 25 inches.

$35 = 5x + 20$

3 lbs required.

38

$v = mt + b$ $f(t) = -32t + 128$

$96 = m + b$

$64 = 2m + b$ Initial Velocity ($t = 0$) is 128.

$32 = -m$

$b = 128$ $0 = -32t + 128$

$32t = 128$

$t = 4$ seconds Velocity is zero

16
7
112

$f(3\frac{1}{2}) = -32 \cdot \frac{7}{2} + 128$

$= -112 + 128$

$= 16$ ft/sec. still rising

39 $v_0 = 1600$ $f(5) = -32$ falling.

$t = 25$ $v = 800$

$v(t) = mt + 1600$

$800 = 25m + 1600$

$-800 = 25m$

$-32 = m$ $v(t) = -32t + 1600$

$v(39) = 640$ slowing but still rising

$v(60) = -320$ moving again, falling

$0 = -32t + 1600$

$t = 50$ sec Bullet stops rising.

Max, min lecture segment

III The general Quadratic Equation:

$y = ax^2 + bx + c$

This is a parabola — put it in graphing form.

use method: completing the square,

$y = a(x^2 + \frac{b}{a}x + ____) + c$

$y = a(x^2 + \frac{b}{a}x + (\frac{b}{2A})^2) + c - (\frac{b}{2A})^2 \cdot a$

$y = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) + c - \frac{b^2}{4A}$

$y = a(x + \frac{b}{2A})^2 + c - \frac{b^2}{4A}$

$y = a(x - (\frac{-b}{2A}))^2 + c - \frac{b^2}{4A}$

↑ Domain cause

Range effect.

~~IV Synthetic substitution.~~

Show of 2 Pg 72

① You have NO control on max or min.

② Label all variables Domain, its etc.

③ Describe "Time" to be maximized or minimized

④ Decide if Question is

A) where does it occur (The max or min)

OR

B) what the max or min actually is.

⑤ Write answer in sentence form.

Lecture 2/16/84

3rd day's lecture

I ~~Pblm 33 pg68~~ Pg 79 #25, 26

II Synthetic Div like Synth Sub.
Zeros of ~~fctns~~. if remainder zero

use same as Demo Yesterday.

$f(x) = 3x^4 - 2x^3 - 10x^2 + 5x - 2$

$f(1) =$

$f(2) =$

as per notes.
show seam factor & remainder theorem.

III Show $y = ax^2 + bx + c \rightarrow y = a\left(x + \frac{b}{2A}\right)^2 + c - \frac{b^3}{4A}$
Max min alg 4 style.

use Eg 2 Pg 72

"You have NO control on "max" or "min"!"
① Label all variables [Draw pictures] the item which is
② Describe, mathematically, the item which is to have the extremum.
③ Decide if the question is
A) when does it occur
or
B) what is the Extremum.
If A then use $\frac{-b}{2A}$
If B then use $c - \frac{b^2}{4A}$.
④ State conclusions in sentence form.

Remember the Parabola

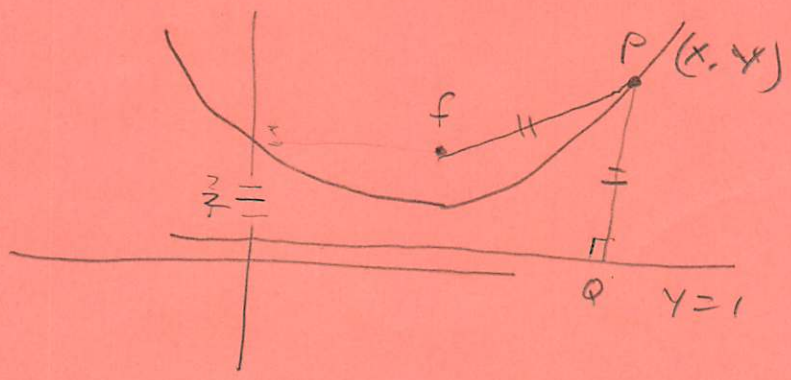
Generic → $y = \frac{1}{4c}(x-h)^2 + k$

basically w/ the $y = mx + b$ is a line.

a parabola has a vertex (the max or min point of the curve) a focus and a directrix

Definition

the Locus of points such that the distance from a point not on the curve called a focus to a point on the curve is equal to the distance from that point on the curve to a fixed line called a directrix.



Example: Given the focus (5, 3) and the directrix $y = 1$, using the distance formula, develop the equation for the parabola.

$\overline{FP} \overset{\text{set}}{=} \overline{PQ}$

$\sqrt{(x-5)^2 + (y-3)^2} = \sqrt{(x-x)^2 + (y-1)^2}$

Keep

Square both sides.

$$(x-5)^2 + (y-3)^2 = (y-1)^2$$

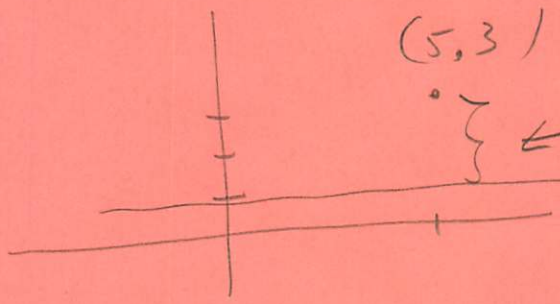
clear these ( )

$$(x-5)^2 + y^2 - 6y + 9 = y^2 - 2y + 1$$ Solve for y

$$4y = (x-5)^2 + 8$$

$$y = \frac{1}{4}(x-5)^2 + 2$$

of course, if we hadn't been required to use the distance formula, we could have "just written down the equation" ...



Halfway between is the Vertex so it is (5, 2)

the distance from the focus to the vertex is $\frac{1}{4c}$ becomes $\frac{1}{4}$

all of this gives us

$$y = \frac{1}{4}(x-5)^2 + 2$$

Had the parabola opened down, c would have been negative.

If the parabola opens Left or Right it is not a function and has equation:

$$x = \frac{1}{4c}(y-k)^2 + h \quad \text{where}$$

(h,k) is still the vertex

if $c < 0$ then the parabola opens left

Now show extension of Prbm 47b.

PHM 47.

Begins on back of this page

(40) $(2,1)$ x int = y int. $\Rightarrow$ slope -1

$y-1 = -(x-2)$

$y = -x+3$

also $y = +\frac{1}{2}x$

$\rightarrow$ if $(0,0)$ otherwise for this reason is method of computing slope. and $\frac{0}{0}$ undefined.

(41) $(-3,1)$ slope $= -1$

$y-1 = (x+3)$

$y = -x-2$

$y = -\frac{1}{3}x$

(42) $y = -\frac{A}{B}x - \frac{C}{B}$

(43) $\left(-\frac{b}{m}, 0\right)$ $\quad y = m\left(x+\frac{b}{m}\right)$

$y = mx+b$

(44) $\frac{x}{a} + \frac{y}{b} = 1$ $\quad$ at $x=0$ for y intercept.

$y = b$

at $y=0$ for x intercept

$x = a$

(45) apply $m =$ formula & $y-y_1 = m(x-x_1)$

Algebraic Version.

$Y = -\frac{A}{B}x - \frac{C}{B}$ ← Given line

⊥ line $m = \frac{B}{A}$ Y intercept 0 ∴

$\overline{OT} \Rightarrow Y = \frac{B}{A}x$

See where two equations intersect

$\frac{B}{A}x \overset{\text{Set}}{=} -\frac{A}{B}x - \frac{C}{B}$

$B^2x = -A^2x - AC$

$x = \frac{-AC}{A^2+B^2}$

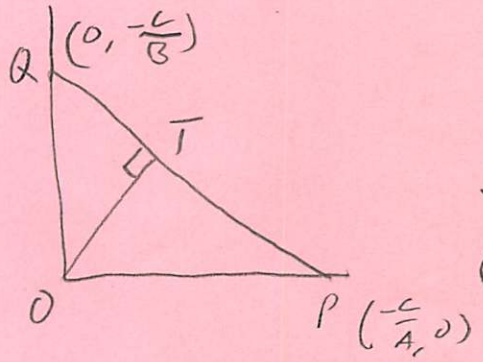
Intersection pt is $\left(\frac{-AC}{A^2+B^2}, \frac{-BC}{A^2+B^2}\right)$

$d = \sqrt{\left(\frac{-AC}{A^2+B^2}\right)^2 + \left(\frac{-BC}{A^2+B^2}\right)^2}$

$= \frac{\sqrt{(AC)^2+(BC)^2}}{A^2+B^2}$

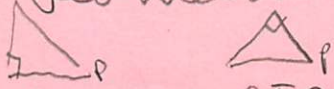
$= \frac{|C|\sqrt{A^2+B^2}}{A^2+B^2} = \frac{|C|}{\sqrt{A^2+B^2}}$

$Ax + By + C = 0$



WTS $OT = \dfrac{|C|}{\sqrt{A^2+B^2}}$

Geometric Version



$\triangle POQ \sim \triangle PTO$ AAA $\angle P$ common

$\dfrac{hyp}{hyp} = \dfrac{OP}{PQ} = \dfrac{OT}{OQ} = \dfrac{\text{side opp } \angle P}{\text{side opp } \angle P}$

so

$$OT = \frac{\overline{OP} \cdot \overline{OQ}}{PQ}$$

$$\overline{PQ} = \sqrt{\left(\frac{C}{A}\right)^2 + \left(\frac{C}{B}\right)^2}$$

$$= |C| \sqrt{\frac{1}{A^2} + \frac{1}{B^2}}$$

$$= |C| \frac{\sqrt{A^2+B^2}}{|AB|}$$

$$= \left|\frac{C}{AB}\right| \sqrt{A^2+B^2}$$

$$= \frac{\left(-\frac{C}{A}\right)\left(-\frac{C}{B}\right)}{\left|\frac{C}{AB}\right| \sqrt{A^2+B^2}}$$

$$= \frac{\frac{C^2}{AB}}{\left|\frac{C}{AB}\right| \sqrt{A^2+B^2}}$$

$$OT = \frac{|C|}{\sqrt{A^2+B^2}}$$

Trig Version on Next Page.

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Given: line with equation Ax + By + C = 0.

Draw the line. The origin is O. The y-intercept is Q. The x-intercept is P. T is a point on PQ.

Want to Prove: The shortest distance from the origin to the line is $OT = \frac{|C|}{\sqrt{A^2 + B^2}}$

Solve the given equation for x-intercept : $\left(-\frac{C}{A}, 0\right)$

Solve the given equation for y-intercept: $\left(0, -\frac{C}{B}\right)$

Geometry version:

1. Draw OT perpendicular to PQ through O. 1. Construction. OT is the shortest distance from O to the line because it is the perpendicular distance.
2. $\Lambda POQ \approx \Lambda PTO$ 2. AAA angle P is common angle, both right triangles
3. $\frac{hyp}{hyp} = \frac{OP}{PQ} = \frac{OT}{OQ} = \frac{side\,opposite\,\angle P}{side\,opposite\,\angle P}$
4. $OT = \frac{OP \bullet OQ}{PQ}$ 4. Multiply by OQ
5. $PQ = \sqrt{\left(\frac{C}{A}\right)^2 + \left(\frac{C}{B}\right)^2}$ 5. Pythagorean Theorem
6. $= |C| \sqrt{\frac{A^2 + B^2}{A^2 B^2}}$ 6. Factor common C out of square root. Combine like terms.
7. $= \left|\frac{C}{AB}\right| \sqrt{A^2 + B^2}$ 7. Definition of absolute value.
8. $= \frac{\left(-\frac{C}{A}\right)\left(-\frac{C}{B}\right)}{\left|\frac{C}{AB}\right| \sqrt{A^2 + B^2}}$ 8. Substitution of lines 5 and 7 into line 4.

9. $= \dfrac{\dfrac{C^2}{AB}}{\left|\dfrac{C}{AB}\right|\sqrt{A^2+B^2}}$ 9. Clear parentheses.

10. $= \dfrac{|C|}{\sqrt{A^2+B^2}}$ 10. Reducing common terms. Thus OT is the shortest length.

Algebraic Version:

1. The given equation simplifies to: $y = -\dfrac{A}{B}x - \dfrac{C}{B}$. The perpendicular line has slope: $\dfrac{B}{A}$ 2. and y-intercept (0, 0). Therefore, $OT \Rightarrow y = \dfrac{B}{A}x$.
3. Where do the two equations intersect?
4. When they are equal: $\dfrac{B}{A}x = -\dfrac{A}{B}x - \dfrac{C}{B}$
5. Multiply by AB to get: $B^2x = -A^2x - AC$
6. Solve for x: $A^2x + B^2x = -AC$
7. $x = \dfrac{-AC}{A^2+B^2}$
8. Substitute for x in line 2 to get $y = \dfrac{-BC}{A^2+B^2}$.
9. coordinates of intersection of the two lines: $\left(\dfrac{-AC}{A^2+B^2}, \dfrac{-BC}{A^2+B^2}\right)$
10. using distance formula from origin to point T: $d = \sqrt{\left(\dfrac{-AC}{A^2+B^2}\right)^2 + \left(\dfrac{-BC}{A^2+B^2}\right)^2}$
11. $d = \dfrac{\sqrt{(AC)^2 + (BC)^2}}{A^2+B^2}$
12. $d = \dfrac{|C|\sqrt{(A)^2+(B)^2}}{A^2+B^2}$ 12. Definition of absolute value: $|x| = \sqrt{(x)^2} = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$
13. $d = \dfrac{|C|}{\sqrt{A^2+B^2}}$ 13. Same as 12. Thus the shortest distance is as stated.

Trigonometric Version:
(see figure 2)

1. $\sin\alpha = \dfrac{OT}{\left|\dfrac{C}{A}\right|}$ $\sin\theta = \dfrac{OT}{\left|\dfrac{C}{B}\right|}$ 1. Definition of sine.

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2. $\dfrac{OT}{\left|\dfrac{C}{B}\right|} = \sin\theta = \sin(90-\alpha)$ 2. $\theta + \alpha = 90^\circ$

3. $\sin(90-\alpha) = \sin 90\cos\alpha - \sin\alpha\cos 90$ 3. sine addition

4. $= \cos\alpha$ 4. $\cos 90 = 0.$

5. $\sin^2\alpha + \cos^2\alpha = 1$ 5. Pythagorean Theorem

6. $\dfrac{(OT)^2}{\dfrac{C^2}{A^2}} + \dfrac{(OT)^2}{\dfrac{C^2}{B^2}} = 1$ 6. Substitute line 1 into line 5.

7. $\dfrac{A^2(OT)^2}{C^2} + \dfrac{B^2(OT)^2}{C^2} = 1$ 7. simplified

8. $(OT)^2\left(\dfrac{A^2}{C^2} + \dfrac{B^2}{C^2}\right) = 1$ 8. Factoring

9. $(OT)^2\left(\dfrac{A^2+B^2}{C^2}\right) = 1$ 9. Simplified.

10. $(OT)^2 = \dfrac{C^2}{A^2+B^2}$ 10. division.

11. $(OT) = \sqrt{\dfrac{C^2}{A^2+B^2}}$ 11. Square root

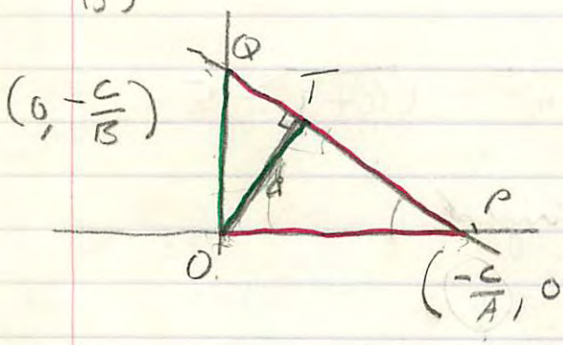
12. $(OT) = \dfrac{|C|}{A^2+B^2}$ 12. Which was to the objective.

Trig Version on Pink Paper

$Ax + By + C = 0$ Geometric Version

~~A) given $A^2+B^2=1$ $\therefore \sqrt{A^2+B^2}=1$ Notice $A, B > 0$~~

B)



WTS $\frac{|C|}{\sqrt{A^2+B^2}} = d$

$\sqrt{\frac{C^2}{A^2} + \frac{C^2}{B^2}} = PQ$

$\frac{\sqrt{C^2(A^2+B^2)}}{AB} = PQ$

$\frac{\frac{-C}{A}}{\frac{|C|\sqrt{A^2+B^2}}{AB}} = \frac{OT}{-\frac{C}{B}}$

$\frac{|C|\sqrt{A^2+B^2}}{AB} = PQ$

$\triangle POQ \sim \triangle PTO$ AAA & P common

$\frac{\text{hyp}}{\text{hyp}} \frac{OP}{PQ} = \frac{OT}{OQ} \frac{\text{side opp P}}{\text{side opp P}}$

$\frac{OT}{OQ} = \frac{OQ}{PQ}$ Solve for OT

$OT = \frac{OQ \cdot OP}{PQ}$

$= \frac{\left|\frac{C}{B}\right| \cdot \left|\frac{C}{A}\right|}{\frac{|C|\sqrt{A^2+B^2}}{AB}} = \frac{\frac{C^2}{AB}}{\frac{|C|\sqrt{A^2+B^2}}{AB}} = \frac{|C|}{\sqrt{A^2+B^2}} = OT$

Example follows.

Given $y = -\frac{4}{3}x + 3$ find shortest distance to origin.

$3y + 4x - 9 = 0$

$\frac{3}{5}y + \frac{4}{5}x - \frac{9}{5} = 0$ Normalized. Then $\left|-\frac{9}{5}\right|$ is that shortest distance.

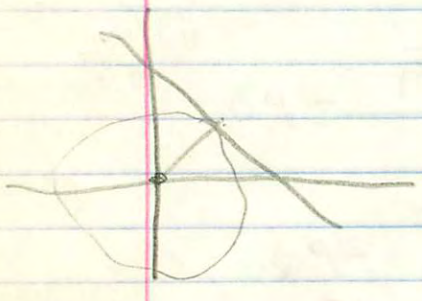
OR

Eq of use of #47

Find the equation of the circle with center at the origin which is tangent to $Y = -\frac{4}{3}x + 3$.

Sol $3Y + 4X - 3 = 0$ $\sqrt{a^2+b^2} = \sqrt{16+9} = 5$

$\frac{3}{5}Y + \frac{4}{5}X - \frac{3}{5} = 0$ Normalized.



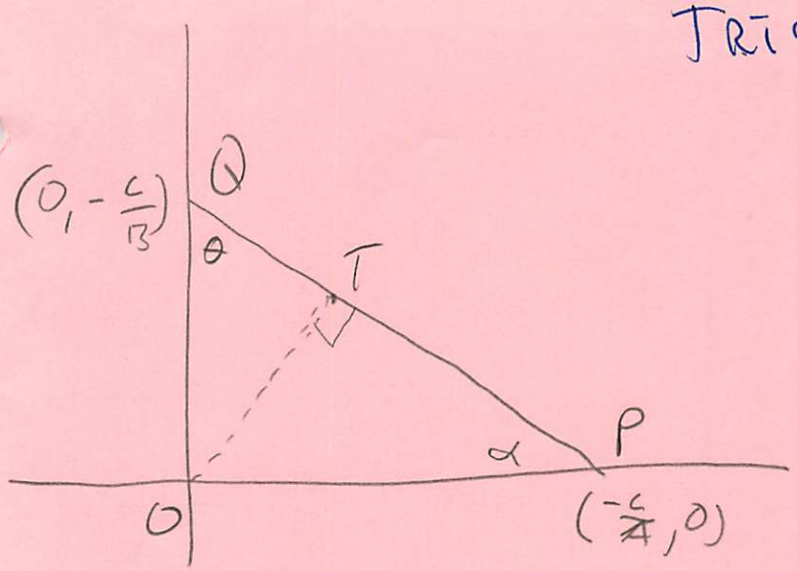
we need a radius. ie the shortest distance.

$\left|\frac{-3}{5}\right|$ = radius

$\therefore$ required equation is

$X^2 + Y^2 = \left(\frac{-3}{5}\right)^2$ $25x^2 + 25y^2 = 9$

TRig Version pg 62 of Pblm. 47b



$\theta + \alpha = 90$

$\theta = 90 - \alpha$

$\sin\alpha = \frac{OT}{|\frac{c}{A}|}$

$\angle P = \alpha$

$\theta = 90 - \alpha$

$\sin\theta = \frac{OT}{|\frac{c}{B}|}$

$\frac{OT}{|\frac{c}{B}|} = \sin\theta = \sin(90-\alpha) = \frac{OT}{|\frac{c}{B}|}$

$= \sin 90\cos\alpha - \sin\alpha\cos 90$

$= \cos\alpha = \frac{OT}{|\frac{c}{B}|}$

$S^2 + C^2 = 1$ so

since $\sin^2\alpha + \cos^2\alpha = 1$

then $\frac{(OT)^2}{\frac{c^2}{A^2}} + \frac{(OT)^2}{\frac{c^2}{B^2}} = 1$

$\frac{A^2(OT)^2}{c^2} + \frac{B^2(OT)^2}{c^2} = 1$

$(OT)^2\left(\frac{A^2+B^2}{c^2}\right) = 1$

$(OT)^2 = \frac{c^2}{A^2+B^2}$

$OT = \frac{|c|}{\sqrt{A^2+B^2}}$

29 $f(x) = 2x$

$g(x) = x+3$

$g(f(x)) = g(2x) = 2x+3$

30 $f(x) = x+3$ $g(x) = 2x$

$g(x+3) = 2x+6$

31 $f(x) \to x-3$ $g(x) = x+3$

$g \circ f = g(x-3) = x$

32 $f(x) \to 2x-1$

$g(x) = 3x+2$

$g(f(x)) = g(2x-1) = 3(2x-1)+2$

$6x-3+2$

$6x-1$

33 $f(x) = 2x-1$ $g(x) = \frac{1}{2}(x+1)$

$g(2x-1) = \frac{1}{2}(2x-1+1) = x$

34. $f(x) \to mx+b$ $m \neq 0$

$y = mx+b$

$x = my+b$

$x-b = my$

$\frac{x-b}{m} = \frac{x}{m} - \frac{b}{m} = f^{-1}(x)$

35. $f(x) = mx+b$ $g(x) = px+q$

$g \circ f = g(mx+b) = p(mx+b)+q = (mp)x + (pb+q)$

36. (1, 5) (2, 7) $\frac{7-5}{2-1} = 2 = m$.

x is # of lbs of weight

a) $y-5 = 2(x-1)$

$y-5 = 2x-2$

$y = 2x+3$

b) $x=0$ $y=3$

c) $8 = 2x+3$

$5 = 2x$

$x = \frac{5}{2}$ lbs.

37 (0, 20) (2, 30) $\frac{20-30}{0-2} = \frac{-10}{-2} = 5$

$y - 20 = 5x$

$y = 5x + 20$

a) $f(x) = 5x + 20$

b) $f(1) = 5 + 20 = 25$ in.

$f(x) = 35 = 5x + 20$

$15 = 5x$

$x = 3$ 3 lbs

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(1, 96) (2, 64) $\frac{96-64}{1-2} = -32$ ft/sec.

$y - 96 = -32t + 32$

c) $f(t)$ $y = -32t + 128$

d) $f(0) = 128$ ft/sec.

e) $f(x) = 0$

$0 = -32t + 128$

$\frac{128}{32} = t$

4 seconds it's at the peak.

d $3\frac{1}{2}$ Balls going up

$f(\frac{7}{2}) = -32(\frac{7}{2}) + 128$

$-112 + 128 = 16$ ft/sec.

e $f(5) = -160 + 128 = -32$ ft/sec. down.

25 $x-2y+3=0$

$x+3=2y$

$\frac{1}{2}x+\frac{3}{2}=y \quad m=\frac{1}{2}$

$y=kx+2$

$k=\frac{1}{2}$ ∥

$k=-2$ ⊥

30 $(k+2)x+(2k+1)y+5=0$

$-(k+2)x-5=(2k+1)y$

$\frac{-(k+2)}{2k+1}=m$

$\frac{-(k+2)}{2k+1}=\frac{1}{2}$ $\qquad$ $\frac{-(k+2)}{2k+1}=-2$

$-2(k+2)=2k+1$ $\qquad$ $-(k+2)=-4k-2$

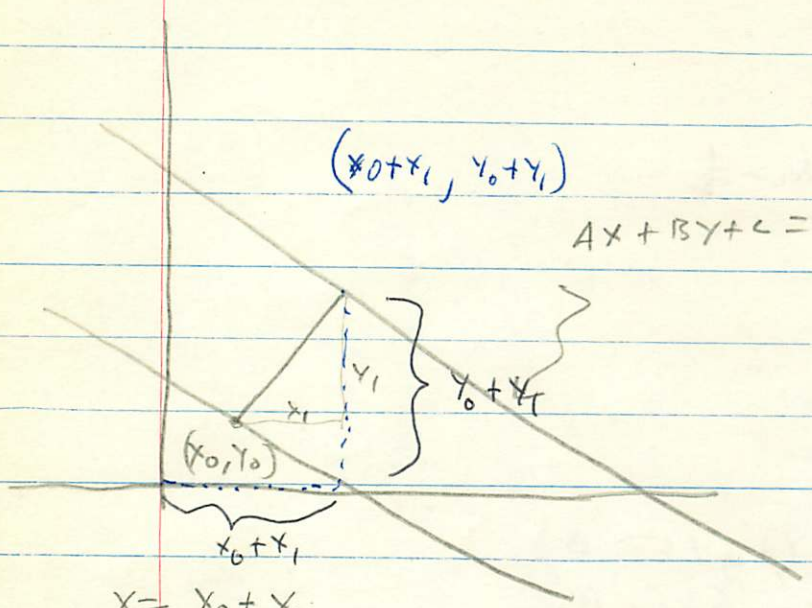
$-2k-4=2k+1$ $\qquad$ $-k-2=-4k-2$

$-5=4k$ $\qquad$ $3k=0$

$-\frac{5}{4}=k$ ∥ $\qquad$ $k=0$ ⊥

31 Put both in $y=mx+b$ form + compare

32 use $m_1=-\frac{1}{m_2}$ for ⊥



$x = x_0 + x_1$

$y = y_0 + y_1$

$Ax + By + C = 0$

$A(x_0+x_1) + B(y_0+y_1) + C = 0$

$Ax_0 + Ax_1 + By_0 + By_1 + C$

$Ax_1 + By_1 + Ax_0 + By_0 + C = 0$

Note (x_0, y_0) is a fixed pt so x_0, y_0 are Known Numbers

Quiz #2

Find the coordinates of the vertex and equation of the directrix of the coordinates of the foci of a parabola which has x intercepts $(-1, 0)$ and $(5, 0)$.

10 minutes.

$(x+1)(x-5) \overset{\text{set}}{=} 0$

$y = x^2 - 4x - 5$

$(x^2 - 4x + 4) - 9$

$y = (x-2)^2 - 9$

$y = \frac{1}{4 \cdot \frac{1}{4}}(x-2)^2 - 9$

Vertex $= (2, -9)$

Focus $\left(2, -\frac{35}{4}\right)$

directrix $y = -\frac{37}{4}$

Everything +

Vertex (2, 0

Focus (2,

( )( ) = 0 ✓

WORK but no conclusion —

Nothing =

10 minutes.

This is only 1 of infinite Parabolas which satisfy requirements.

For example: $\frac{1}{4}(\quad x \quad) \overset{\text{set}}{=} 0$

$y = \frac{1}{4}(x-2)^2 - \frac{9}{4}$ here $\left(2, -\frac{9}{4}\right)$ vertex.

$c = 1$

$\left(2, -\frac{5}{4}\right)$ focus

$y = -\frac{13}{4}$ dir.

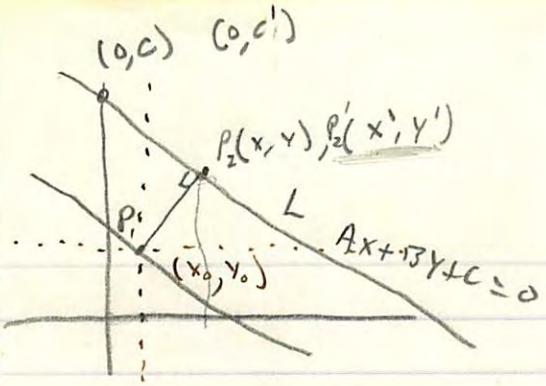
Analysis
El Quizzzo
Mr. Mumaugh

Name _____

Per _____

Given: $f(x) = -\frac{1}{2}x^2 - x + 4.$

Compute the equation **for each of the *three* lines** which pass through the focus of f *AND* through an axis intercept of f.



$Ax' + By' + \mathbb{C} = 0$

$x = x_0 + x' \qquad y = y_0 + y' \qquad C \to C'$

$\therefore Ax + By + C_1 = 0$ The transformed axis becomes:

$$A(x_0 + x') + B(y_0 + y') + C' = 0$$

$$Ax_0 + By_0 + Ax' + By' + C' = 0$$

when $\mathbb{C} = Ax_0 + By_0 + C'$ we have:

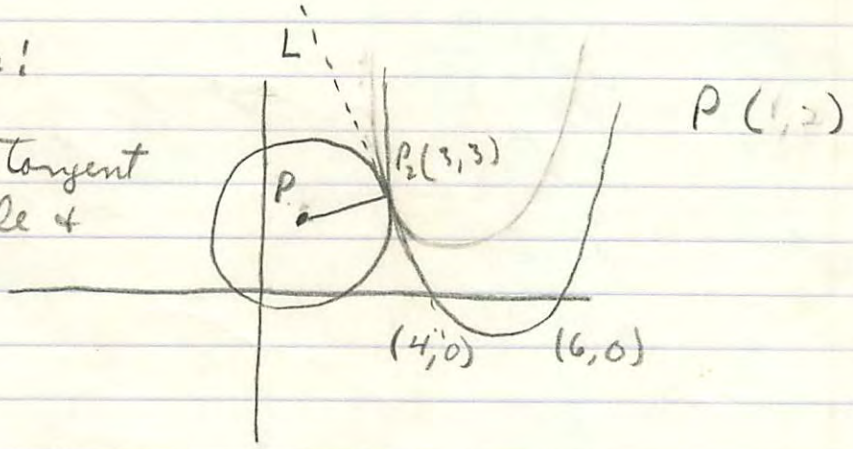
$$Ax' + By' + \mathbb{C} = 0$$

By Problem 47 b page 62, we know the distance from the origin to a line is:

$$\frac{|\mathbb{C}|}{\sqrt{A^2+B^2}} = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2+B^2}}$$

Possible use!

Assume L is tangent to both the circle & Parabola.



$(x-4)(x-6) = 0$

$x^2 - 10x + 24 = 0$

$2x - 10 = f'(x)$

$f'(3) = -4$

$f(3) = 9 - 30 + 24 = 3$

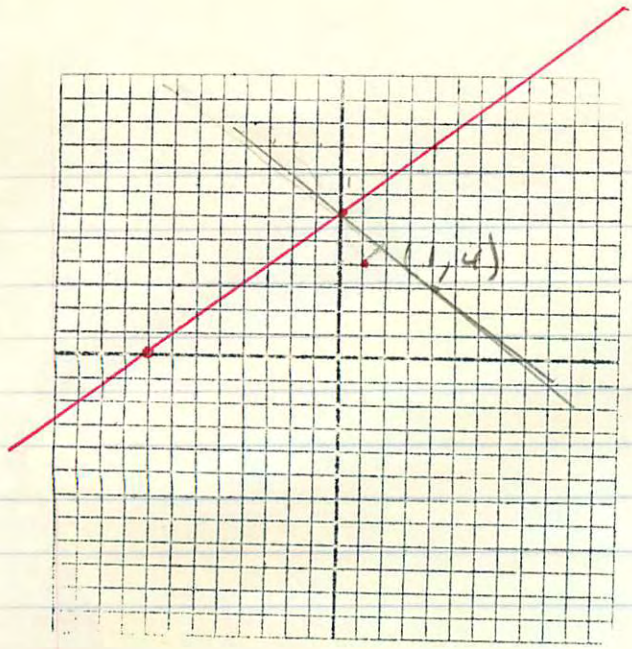
Equat of Tangent line is: $Y - 3 = -4(x-3)$

$Y - 3 = -4x + 12$

$Y = -4x + 15$

$4x + Y - 15 = 0.$

$Y - Y_1 = \frac{1}{4}(x - X_1)$



$y = \frac{3}{4}x + 6$

$-3x + 4y - 24 = 0$

using Pg 68 #33

$x_0 = 1$

$y_0 = 4$

Dist fm (1, 4) to line:

$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$\frac{|-3 + 16 - 24|}{5} = \frac{|-11|}{5} = \frac{11}{5}$$

$$(x-1)^2 + (y-4)^2 = \frac{121}{25}$$

425

$$x^2 - 2x + 1 + y^2 - 8y + 16 = \frac{121}{25}$$

$$x^2 + y^2 - 2x - 8y + 17 = \frac{121}{25}$$

$$25x^2 + 25y^2 - 50x - 200y + 304 = 0$$

17
25
235
14
5
425
-121
304

Find the equation of the circle tangent to f(x) w/ center at (1, 4) if (0, 6) and (−8, 0) satisfy f(x). Write final equation in the form $Ax^2 + By^2 + Cx + Dy + E = 0$

the line

#1 hint only 2 coefficients in the following are incorrect.

$25x^2 + 25y^2 + 50x + 200y + 304 = 0$

#2 hint See page 681

I $m = 4$ $(-5, 6)$ $y = 4x + 26$

II $(1, 3.0)$ $(-1, 2.3)$ $y = \frac{7}{2}x + \frac{53}{2}$

III $m = \frac{3}{2}$ x-int $= \frac{2}{3}$ $y = \frac{3}{2}x - 1$

10 min Quiz

Before lecture on Extreme value ph/ms

and alg & Geo version of ph/m #47

Thm 3

Look at the equation $y = x^2$.
It has a minimum at (0,0).
Look at $y = -x^2$
It has a maximum at (0,0)
Look at $y = x^2 + 2$
min at (0,2)
Consider $y = (x+1)^2 + 2$
Notice if $x = -1$ $y = 2$
if $x = -2$ $y = 3$
if $x = 0$ $y = 3$
it appears that at $x = -1$ we have a minimum. Thus min coord would be (-1, 2)
$f(x) = x^2 + 2x + 3$ what are coordin. of max or min.?

$f(x) - 3 = x^2 + 2x$
$f(x) - 3 + 1 = x^2 + 2x + 1$
$f(x) - 2 = (x+1)^2$
$f(x) = (x+1)^2 - 2$ Same equation as ob
Thm 3 pg 72 works this out for the general case $f(x) = ax^2 + bx + c$.
Prove thm 3 using the method of completing the square.

(21) $25 + 15t - 4t^2 = 0$

$$t = \frac{-15 \pm \sqrt{225 + 400}}{-8}$$

$$t = \frac{-15 \pm \sqrt{625}}{-8}$$

$$\frac{15 \mp 25}{8}$$ 5 sec.

$c - \frac{b^2}{4a} = \text{max.}$ axis of sym. $x = \frac{b}{2a}$

11) $f(x) = x^2$ min $(0,0)$

14) $f(x) = x^2 + 4x + 2$ $\frac{-4}{2} = -2$

$2 - \frac{16}{4} = 2 - 4 = -2$ min.

16) $f(x) = 2 - x - x^2$ $\frac{-1}{2}$

$2 + \frac{1}{4} = \frac{9}{4}$ max.

21 $h(t) = 100 + 60t - 16t^2$

$-100 = 100 + 60t - 16t^2$ $16t^2 - 60t - 100 = 0$

$t = \frac{-60 \pm \sqrt{3600 + 6400}}{-32}$

$t = \frac{-60 \pm 100}{-32}$

$t = \frac{-160}{-32} = 5$ sec

$-100 = 100 + 60t - 16t^2$

$0 = 200 + 60t - 16t^2$

$0 = 150 + 15t - 4t^2$

$\frac{-15 \pm \sqrt{225 + 2400}}{-8}$

$\frac{1600}{3600}$ $\frac{250}{2}$

$100 + 25$

$-16t^2 + 60t = 0$

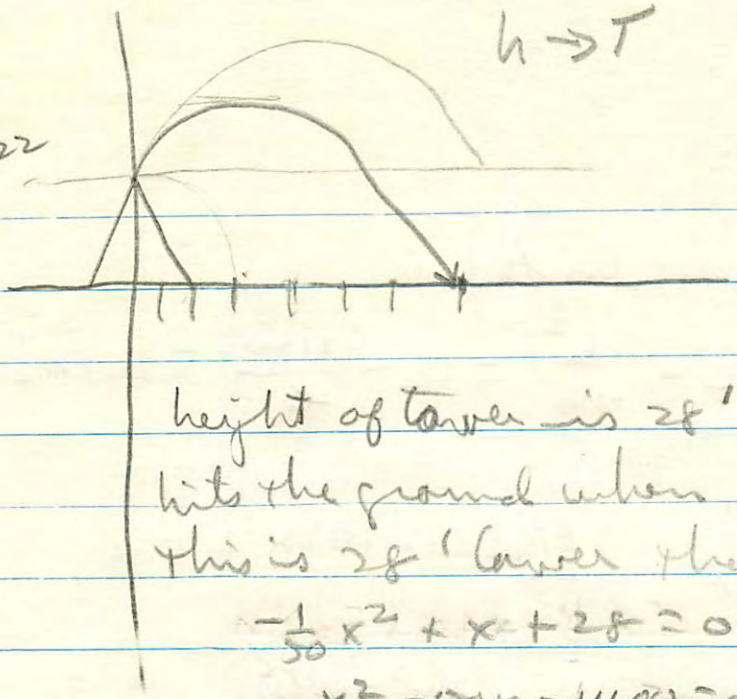
$-16t = -60$

$t = +\frac{60}{32} = +\frac{15}{8}$

$c - \frac{b^2}{4A}$

$h \Rightarrow T$

#22



$V = 28 + x - \frac{1}{50}x^2$

height of tower is 28' (let $x=0$ is initial pt)
hits the ground when $f(x) = 0$. since
this is 28' taller than intial pt.

$$-\frac{1}{50}x^2 + x + 28 = 0$$

$$x^2 - 50x - 1400 = 0$$

$$(x-70)(x+20)$$

$x = 70$ that is 70 ft from starting pt.

$f(x) = -\frac{1}{50}x^2 + x + 28$

$Y \Rightarrow c - \frac{b^2}{4a}$

$28 + \frac{1}{\frac{2}{25}}$

$x \Rightarrow \frac{-b}{2a} = \frac{-1}{-\frac{2}{50}}$

$28 + \frac{25}{2} = 40\frac{1}{2}$

$(25, 40\frac{1}{2})$ coord of highest pt.

74

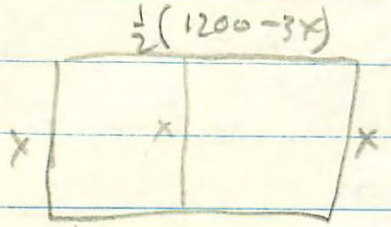
(23)

$A(x) = 90x - 2x^2$

$\frac{-90}{-4} = 22\frac{1}{2}$ length side

Greatest area $c - \frac{b^2}{4a} = 0 - \frac{8100}{-8} = 1012\frac{1}{2}$ sq ft.

(25)



$\frac{1}{2}(1200-3x)x = A(x)$

$600x - \frac{3}{2}x^2 = A(x)$

$\frac{-600}{-3} = 200 = x$

$\frac{1}{2}(1200-600) = \frac{1}{2}(600) = 300$

max area (Not required) $\Rightarrow \frac{360000}{6} =$

60,000 sq ft.

(26) x is 1st Number

$12-x$ is 2nd number

$12x - x^2 = f(x)$

$\frac{144}{4} = 36$ is greatest product.

$\frac{-12}{2} = 6$ Both #s are 6.

(27) x is 1st

$12-x$ is 2nd.

$144 - \frac{576}{8}$

$144 - 72 = 72$

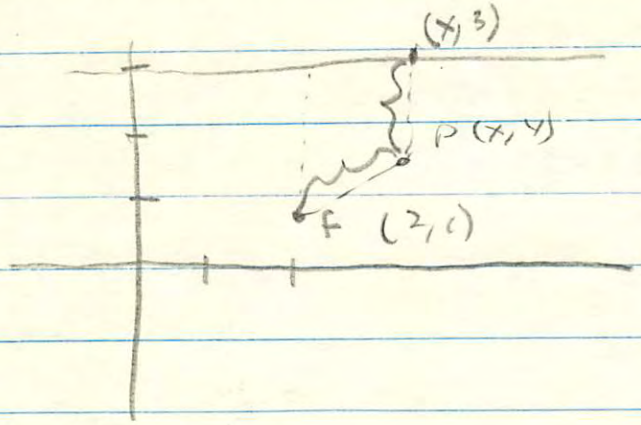
$x^2 + (12-x)^2 = f(x)$

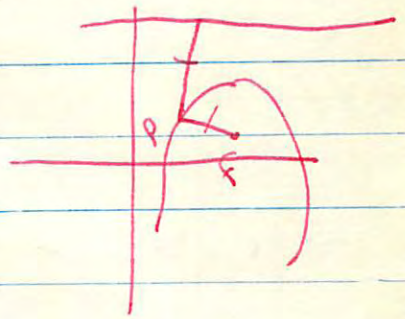
$x^2 + 144 - 24x + x^2 = f(x)$

$2x^2 - 24x + 144 = f(x)$

Draw the parabola described

focus (2,1) & D: $y=3$





P(x,y) By Def $\sqrt{(x-2)^2+(y-1)^2} = \sqrt{(x-x)^2+(y-3)^2}$

$(x-2)^2+(y-1)^2 = (y-3)^2$

$x^2-4x+4+y^2-2y+1 = y^2-6y+9$

$x^2-4x-4 = -4y$

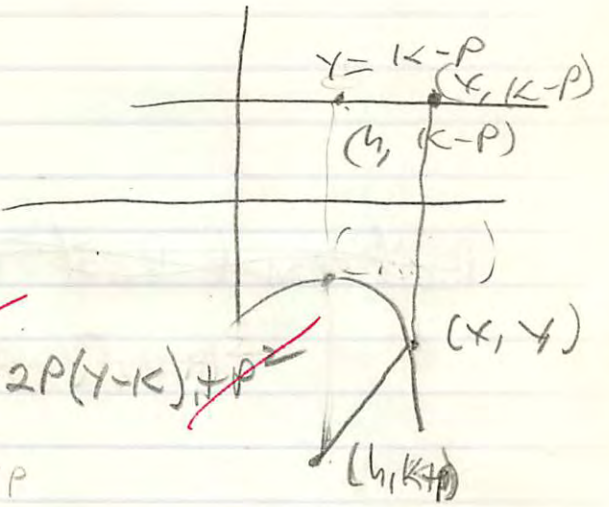
$\frac{1}{4}x^2+x+1 = y$ is the equation of the line that satisfies the condition called a parabola.

25

Vertex (h,k) $\quad y=\delta$ directrix

wTS: $(x-h)^2 = 4p(y-k)$

Focus $(h, k+p)$



$$\sqrt{(x-h)^2+((y-k)+p)^2} = \sqrt{(y-(k+p))^2}$$

$$(x-h)^2 + (y-k)^2 - 2p(y-k) + p^2 = (y-k)^2 + 2p(y-k) + p^2$$

$$(x-h)^2 = 4p(y-k) \qquad 4py - 4kp$$

The key to this problem is regrouping.

$$y-(k+p) \Rightarrow (y-k)-p$$

and $$y-(k-p) \Rightarrow (y-k)+p$$

(26)

$y = ax^2+bx+c \quad a \neq 0 \Rightarrow (x-h)^2 = 4p(y-k)$.

$y = a\left(x-\frac{b}{2a}\right)^2 + \left(c-\frac{b^2}{4a}\right) \quad h = \frac{b}{2a} \quad k = c-\frac{b^2}{4a}$.

$y - c + \frac{b^2}{4a} = a\left(x-\frac{b}{2a}\right)^2$

$\left(y-\left(\frac{4ac+b^2}{4a}\right)\right) = a\left(x-\frac{b}{2}\right)^2$

① complete sq.
② compare

$\frac{1}{a}\left(y-\left(\frac{4ac-b^2}{4a}\right)\right) = \left(x-\frac{b}{2a}\right)^2$

$4p = \frac{1}{a} \Rightarrow p = \frac{1}{4a}$.

eg. 1 $y = 6x^2+7x+8$

$y = 12x+7 \qquad -\frac{7}{12} \qquad 8-\frac{49}{24} \qquad \frac{192}{24}-\frac{49}{24} = \frac{143}{24}$

$\left(x-\frac{7}{12}\right)^2 = 6\left(y-\frac{143}{24}\right)$

$6 = \frac{1}{4p}$

$p = \frac{1}{24}$.

Synthetic Division
Lecture
Prior to
8.4, 9.2

divide $x+2 \overline{)3x^4-2x^3-10x^2+5x-2}$

$3x^3-8x^2+6x-7$

$3x^4+6x^3$

$-8x^3$

$-8x^3-16x^2$

$6x^2$

$6x^2+12x$

$-7x$

$-7x-14$

12

$f(-2)$

$$\begin{array}{r} 2679 \\ 12\overline{)32152} \\ 24 \\ \hline 81 \\ 72 \\ \hline 95 \\ 84 \\ \hline 112 \\ 108 \\ \hline 4 \end{array}$$

$3x^4-2x^3-10x^2+5x-2 = (x+2)(3x^3-8x^2+6x-7)-12$

$(x-r)$ $Q(x)$ remainder

$32152 = (12)(2679)+4$

Consider

	3	-2	-10	5	-2
-2	3	-8	6	-7	12
			$Q(x)$		Remainder

$=(x-r)$

This is called Synthetic Division

TRY dividing by $x-2$

	3	-2	-10	5	-2
2	3	4	-2	1	0

So $(x-2)(3x^3+4x^2-2x+c)+0$

↑ No remainder!

There are many other uses for this same thing.

$f(x) = 3x^4 - 2x^3 - 10x^2 + 5x - 2$

$f(0) = ?$
$f(1) = ?$
$f(2) = ?$
$f(-1) = ?$
$f(-2) = ?$

Now Notice

	3	-2	-10	5	-2
0 |					-2
1 |	3	1	-9	-4	-6
2 |	3	4	-2	1	0
-1 |	3	-5	-5	10	-12
-2 |	3	-8	6	-7	12

This is called synthetic substitution.

Summary "is a factor of" = Divides w/ Zero Remainder

Turn more page ↓

$P(x) = (x-a) \cdot Q(x) + R(x)$ Remainder Theorem

How many twos are in twelve?

2 divided into 12

$$2\overline{)12} \qquad 6 \qquad \qquad 2\overline{)6} \qquad 3$$

That is 2 divides evenly into 12 How many times

$2\overline{)12}$

$2\overline{)12}$

$2\overline{)12}$

$\vdots$

$2\overline{)12}$

} Now Then 2 times.

what's the problem? after dividing in to P(x) you then must divide in to Q(x). Not P(x) again.

Then Go to

x^2+6x+9

x^2-x-2

$-2x^2-12x-18$

$-x^3-6x^2-9x$

$x^4+6x^3+9x^2$

$x^4+5x^3+x^2-21x-18$

Show $x+3$ is a multiple factor of

	1	5	1	-21	-18
-3	1	2	-5	-6	0
-3	1	-1	-2	0	

END of lecture

$\quad\quad\quad

$f(2) \Rightarrow$ 2 | 3 −2 −10 +5 −2 this means
6 8 −4 2
3 4 −2 −1 0

$(x-2)(3x^3+4x^2-2x-1)+f(2)=f(x)$.

that is given $f(x)$ we can rewrite $f(x)$

thusly $f(x)=g(x)h(x)+J(x)$

where $f(x)=3x^4-2x^3-10x^2+5x-2$
$g(x)=(x-2)$
$h(x)=(3x^3+4x^2-2x-1)$
$J(x)=f(2)$

then if $f(r)=0$ then $x-r$ is a factor.

~~If we had we would say~~

can also be interpreted as saying

the Equation: $3x^4-2x^3-10x^2+5x-2=0$

has solution r.

$P(x) = S(x)Q(x) + R(x)$

7 (8) $P(x) = 4x^3 - 3x^2 + 2x - 2$

$S(x) = x^2 + x + 5$

$$\begin{array}{r|l} & 4x - 7 \\ \hline x^2+x+5 & 4x^3 - 3x^2 + 2x - 2 \\ & 4x^3 + 4x^2 + 20x \\ \hline & -7x^2 - 18x - 2 \\ & -7x^2 - 7x - 35 \\ \hline & -11x + 33 \end{array}$$

$P(x) = S(x)(4x-7) + (-11x+33)$

8 4

$P(x) = x^4 + 4x - 3 \qquad S(x) = 2 - x + 4x^2$

$$\begin{array}{r|l} & \frac{1}{4}x^2 + \frac{1}{16}x + \frac{1}{64} \\ \hline 4x^2 - x + 2 & x^4 + \quad 4x - 3 \\ & x^4 - \frac{1}{4}x^3 + \frac{1}{2}x^2 \\ \hline & \frac{1}{4}x^3 + \frac{7}{2}x \\ & \frac{1}{4}x^3 - \frac{1}{16}x^2 + \frac{1}{8}x \end{array}$$

9 $P(x) = x^3 - x + 1 \qquad S(x) = x^3 + x + 1$

$$\begin{array}{r|l} & 1 \quad \frac{1}{16}x^2 + \frac{27}{8}x \\ \hline x^3+x+1 & x^3 - x + 1 \\ & x^3 + x + 1 \\ \hline & -2x \end{array}$$

11 x^3-3) x^4+2x^3-3x-6 $\quad x+2$

$x^4 \quad -3x$

$2x^3-6$

$2x^3-6$

0

Yes it is a factor.

14 $(x^3+2x^2+3x+4 \mid \div(x+3)$

-3 | $1 \quad 2 \quad 3 \quad 4$

$\quad\quad -3 \quad -18$

$\quad 1 \quad -1 \quad 6 \quad -14$

$Q(x) = x^2-x+6$

$R(x) = -14$

15 -2 | $1 \quad 0 \quad 0 \quad 0 \quad 2$

$\quad\quad -2 \quad 4 \quad -8 \quad 16$

$\quad 1 \quad -2 \quad\quad\quad 18$

$Q(x) = x^3-2x^2+4x-8$

$R(x) = 18$

16 1 | $1 \quad 0 \quad -4 \quad 2 \quad 1$

$\quad\quad 1 \quad 1 \quad -3 \quad -1$

$Q(x) = x^3+x^2-3x-1$

$R(x) = 0$

17 2 | $1 \quad 5 \quad -2 \quad a \quad -7$

$\quad\quad 2 \quad 14 \quad 24 \quad 2a+48$

$\quad 1 \quad 7 \quad 12 \quad a+24$

$2a+48-7=0$

$2a = -41$

$a = -\frac{41}{2}$

18 -2 | 1 0 0 0 −2 a

1 −2 4 −8 14 $\quad -28$

$a = 28$

19 3 | 1 −1 4 1

1 +2 10 31

1 5 25

1 8

1

$(x-3)^3 + 8(x-3)^2 + 25(x-3) + 31$

20 -2 | 1 5 7 0 −3

1 3 1 −2 1

1 1 −1 0

1 −1 1

1 −3

1

$(x+2)^4 - 3(x+2)^3 + (x+2)^2 + 1$

85

21

$P(x)$
$S(x)$
$T(x)$

No $R(x)$ since P is a factor of S etc

$S(x) = P(x) Q_1(x)$

$T(x) = S(x) Q_2(x)$

$T(x) = P(x) Q_1(x) Q_2(x)$

$T(x) = P(x) Q_3(x)$ where $Q_1(x) Q_2(x) = Q_3(x)$

thus by def $P(x)$ is a factor of $T(x)$

22 Given $S(x) = P(x) Q_1(x)$

$S(x) T(x) = P(x) Q_1(x) T(x)$

Since there is No $R(x)$, $P(x)$ is a factor.

23

Given $S(x) = P(x) Q_1(x)$
$T(x) = P(x) Q_2(x)$

$S(x) + T(x) = P(x) Q_1(x) + P(x) Q_2(x)$
$= P(x) [Q_1(x) + Q_2(x)]$

let $Q_1(x) + Q_2(x) = Q_3(x)$

$S(x) + T(x) = P(x) Q_3(x)$

again No $R(x)$ $\therefore$ $P(x)$ is a factor of $S(x) + T(x)$.

if $S(x) + T(x)$ is the zero poly then we have the trivial solution.

2-6 Remainder Thm.

Use Synthetic substitution to find the value, as illustrated:

$P(x) = (x-a)Q(x) + R$ let $x = a$

$P(a) = (a-a)Q(x) + R$

$P(a) = R$ (R is a constant.)

$P(x) = x^3 + 2x^2 - 5x + 1$

1) $P(-3) \Rightarrow$ -3 | $1 \quad 2 \quad -5 \quad 1$

$1 \quad -1 \quad -2 \quad 7 \qquad P(3) = 7$

2) $P(0) = 1$

3) $P(1) = 1 + 2 - 5 + 1 = -1$

4) $P(-1) = -1 + 2 + 5 + 1 = 7$

13) $u(x) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc$

a) $1 \quad -(a+b+c) \quad (ab+bc+ca) \quad -abc$

$a \quad -ab - ca \quad abc$

$1 \quad -b-c \quad bc \quad 0 = u(a)$

14)

$b \quad b \quad -ab - bc \quad abc$

$1 \quad -a-c \quad ca \quad 0 = u(b)$

15) $c \quad -cb - ac \quad abc$

$1 \quad -b-a \quad bc \quad 0 = u(c)$

16) $0 \quad 1 \quad 0 \quad 0$

$-a-b-c \quad ab+bc+ca \quad -abc$

27. ie show $P(x_1) = P(-x_1)$

$P(x) = ax^6 + bx^4 + cx^2 + d$

x_1 | a 0 b 0 c 0 d

a $ax_1 + b$ $ax_1^2 + bx_1 + c$ $ax_1^3 + bx_1^2 + cx_1 + d$

$-x$ | a 0 b 0 c 0 d

a $-ax$ $ax^2 + b$ etc. they are the same since the 0's make the product 0.

24 ie Show $P(x) = -P(-x)$

$ax^7 + bx^5 + cx^3 + dx$

x | a 0 b 0 c 0 d 0

a ax $ax^2 + b$ $ax^3 + bx + c$

ax $ax^2 + b$ $ax^3 + b$

Pg 89-97

Lecture

factor thm states if $x-a$ is a factor of $f(x)$ **Then** $f(a)=0$.

Multiplicity: using depressed equations - if a root works (e.g.) 3 times Then ~~it to~~ The original equation has that root with a multiplicity of 3. e.g.!

for comparison $2\sqrt{12}$ $\overset{6}{}$ now $2\sqrt{12}$ again? No!
$2\sqrt{6}$ ← depressed equations

$x+1$ | x^2+2x+1
x^2+2x+1
x^2+2x+1
$2x^3+4x^2+2x$
$x^4+2x^3+x^2$
$x^4+4x^3+6x^2+4x+1$

$f(x) = x^4+4x^3+6x^2+4x+1$

	1	4	6	4	1
-1	1	3	3	1	0
-1	1	2	1	0	
-1	1	1	0		
-1	1	0			

$\therefore$ 4 are ≥ 2 would: True original two times depressed equation. $\therefore$ equation.

-1 has multiplicity 4.
$\therefore (x+1)$ is a factor 4 times.

Demonstrate $\frac{P}{Q}$ $(ax-b)(cx-d)=0$

$$acx^2-(ad+bc)x+bd=0$$

$$\left(x-\frac{b}{a}\right)\left(x-\frac{d}{c}\right)=0$$

roots $\frac{b}{a}$ $\frac{d}{c}$

Notice Product $\frac{bd}{ac}$

Leading Coefficient
Denom of
is at least a factor of Leading coefficient
Numerator is at least a factor of Last coefficient

If Solves, writing
Q ≠ 0 Then

Solve for all ~~roots~~.

$3x^3 - 2x^2 + 2x + 1 = 0$

$\frac{P}{Q} \Rightarrow \pm\left\{\frac{1}{1}, \frac{1}{3}\right\}$

$\frac{1}{1} \quad \frac{-1}{1} \quad \frac{1}{3} \quad \frac{-1}{3}$

use Synthetic division

Note Students are welcome to try any #s however when they get tired of Not Guessing correctly they can use the results of the Rational Root Thm.

	3	-2	2	1
1	3	1	3	4
-1	3	-5	7	-6
$\frac{1}{3}$	3			
$-\frac{1}{3}$	3	-3	3	0

$-1 \quad -1 \quad 1$

$x^2 - x + 1$

$x = \frac{1 \pm \sqrt{1-4}}{2}$

$x = \frac{1 \pm i\sqrt{3}}{2}$

continue to next 2 pgs

Continuing . . .

Descartes' Rule of Sign Variations

From struggling through the last section and trying to find Roots. It should be clear that any rule that will yield either the Number of Positive or Negative roots or yield at least an upper + lower bound would be a time saver.

As it turns out we have both in Descartes' Rule of Variation of Signs. Read Pages 98–100, then discuss.

eg 2 pg 99.

$$P(x) = x^7 - x^5 + 2x^2 + 1$$

1 1 0 $\Rightarrow$ 2 Positive Real Roots

$$P(-x) = -x^7 + x^5 + 2x^2 + 1$$

1 0 0 $\Rightarrow$ 1 Negative Real Root.

Note: not necessarily rational roots

Now we know how many of each, let us see what bounds we can establish.

	1	0	−1	0	0	2	0	1	
1	1	1	0	0	0	2	2	3	upper bound all >0
−1	1	−1	0						No got a 0
−2	1	−2	3	−6	12	−22	44	−89	Lower alt.

eg 3 pg 100 $2x^3 - 5x^2 + x + 9 = 0$

	2	−5	1	9		
1	2	−3			NO change of Sign	
2	2	−1			"	
3	2	1	4	21	upper Bound	
−1	2	−7	8	1	NO 2 +'s	
−2	2	−9	17	−25	Lower Bound alternate Signs.	

Discuss Oral exercise pg 101

Decarte's Rule of Signs.

① $f(x)$ ⇒ # of variations of signs = # of Positive Real Roots or fewer (by even number) eg. 8 changes ⇒ 8, 6, 4, 2, 0 positive Real Roots

② $f(-x)$ ⇒ # of variations = # of Neg Roots or fewer by even number. eg. 5 changes ⇒ 5, 3, 1 Neg Roots.

③ $f(a)$ ⇒ ALL NON Negative coefficients ⇒ a is upper bound of all Real Roots

④ $f(a)$ ⇒ ALTernately coefficients ⇒ a is a lowerbound of all Real Roots.

Furthermore

If f has all rational coefficients,

then if there are ANY irrational or ANY complex roots,

they will be in conjugate Pairs.

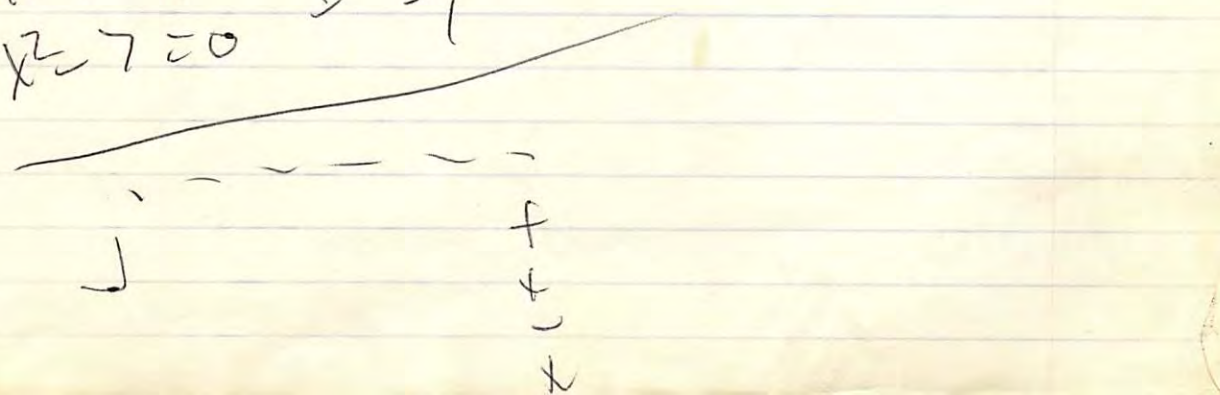
$\sqrt{7}$

$x = \sqrt{7}$

$x^2 = 7$

$x^2 - 7 = 0$

$\pm \frac{1, 7}{1}$



$x^3 - x^2 - 8x + 12$, $x - 2$

9

2 | 1 −1 −8 12

1	1	−6	0
1	3	0	

multp of 2.

10 $x^4 + 2x^3 - 2x - 1$ $x + 1$

−1	1	2	0	−2	−1
				1	1
	1	1	−1	−1	0
			0		
	1	0	−1	0	
		1	−1	0	
		1	−2		

multp of 3

(11) $x^4 - 6x^2 + 9$ $x + \sqrt{3}$

$-\sqrt{3}$	1	0	−6	0	9
		$-\sqrt{3}$	3	$3\sqrt{3}$	−9
	1	$-\sqrt{3}$	−3	$3\sqrt{3}$	0
			6	$-3\sqrt{3}$	
	1	$-2\sqrt{3}$	−3	0	
		$-\sqrt{3}$	$3\sqrt{3}$		
	1	$-3\sqrt{3}$			

multp of 2

(13) $x^3 + 3x^2 + cx + d \quad (x+2)^2$

	1	3	c	d	
-2			-2	$-2c+4$	
	1	1	$c-2$	$-2c+4+d$	$-2c+4+d=0$
		-2	2		
	1	-1	c		$c=0$
					$d=-4$

① $2x^3-3x^2-x+12=0$ $\quad \frac{p}{q} \Rightarrow \pm\{\frac{1\ 2\ 3\ 4\ 6\ 12}{1,2}$

	2	−3	−1	12
−1	2	−5	4	8
−2	2	−7	13	−
$-\frac{3}{2}$	2	−6	8	0

} root between

$x \in \{-\frac{3}{2}, \frac{3\pm i\sqrt{7}}{2}\}$

$2x^2-6x+8=0$

$x^2-3x+4=0$

$\frac{3\pm\sqrt{9-16}}{2}$ $\quad \frac{3\pm i\sqrt{7}}{2}$

② $3x^3-10x^2+7x+10=0$ $\quad \frac{1,2,5,10}{1,3}$

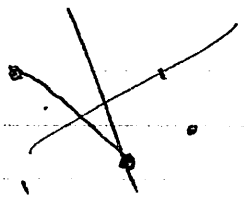
	3	−10	7	10
−1	3	−13	20	−10 min.
$-\frac{1}{3}$	3	−11		
$\frac{1}{3}$	3	−9	4	
$-\frac{2}{3}$	3	−12	15	0

$x \in \{-\frac{2}{3}, 2\pm i\}$

$3x^2-12x+15=0$

$x^2-4x+5=0$

$\frac{4\pm\sqrt{16-20}}{2}$ $\quad x = \frac{4\pm 2i}{2}$ $\quad 2\pm i$.



③ on Nix paper.

④ x^5+x^4-2x-3

1 + 2 0 −

+ + − −

No Rat Roots.

I cannot determine any roots

	1	1	0	0	−2	−3
−1	+	0	0	0	−2	1
1	1	2	2	2	0	−3
−3	3	−2	6	−18	+	—
3	1	4	12	36		

(3) $x^4 - 3x^2 + 2x + 2$ $\{\pm 1 \pm 2 \pm \frac{1}{2}\}$

1	1	−3	2	2
	1	−2	0	NO
4	1	−4	6	NO
2	1	−1	0	NO
−2	1	−5	12	NO
$\frac{1}{2}$	1	$-\frac{5}{2}$	$\frac{3}{4}$	NO
$-\frac{1}{2}$	1	$\frac{7}{2}$	$\frac{1}{4}$	NO

No rational roots.

(5) $x^3 - \frac{4}{3}x^2 - x + \frac{7}{3} = 1$

$3x^3 - 4x^2 - 3x + 4 = 0$ $\{\pm 1 \pm \frac{1}{3} \pm 4 \pm 2 \pm \frac{4}{3} \pm \frac{2}{3}\}$

1	3	−4	−3	4
	3	−1	−4	0
1	3	−2	−6	
−1	3	−4	0	
$\frac{4}{3}$	3	4 0		

1 is a root, multiple of 1

−1 is a root multiple of 1

$\frac{4}{3}$ is a root.

(13) Students are to fill in ALL values. Use for comparison w/ Descartes' rule next section

$x^3 - 4x^2 - 4x + 17 = 0$

	1	−4 −4 17
−3		− } between −3 & −2
−2		+ }
−1	1	+
0		+
1		+
2		+ } between 2 & 3
3		− }
4		+ } between 3 & 4
5	all +	+

↳ 3

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14 $x^3 - 4x^2 - 4x + 15 = 0$

	1	−4	−4	15	
−3					−
−2					− }
−1					+ }
0					+
1					+
2					+ }
3	1				−
4	1	0			− }
5	1				+

17 If leading coeffic is 1 then $\frac{p}{q} \Rightarrow \frac{p}{1}$ which is p is integer.

18 If Last coeffic is 1 then $\frac{p}{q} \Rightarrow \frac{1}{q}$ which is recip. of integer q.

9 multiply then only Rat possible roots are ± 1 ± 5

Upper Bound is a Non negative integer

Lower Bound is a Non positive integer

5. $x^3 - \frac{4}{3}x^2 - x + \frac{7}{3} = 1$

$3x^3 - 4x^2 - 3x + 4 = 0$

	3	−4	−3	4	
−1	3	−7	4	0	$x \in \{-1, \frac{4}{3}, 1\}$

$(3x - 4)(x - 1)$

⑥ $x^3 + \frac{3}{4}x^2 - 4x - \frac{1}{2} = \frac{5}{2}$

$4x^3 + 3x^2 - 16x - 2 = 10$ 1, 2, 3, 4, 6, 12

$4x^3 + 3x^2 - 16x - 12 = 0$ 1, 2, 4

	4	3	−16	−12
−1	4	−1	−17	5
1	4	7	−12	−24
$\frac{1}{4}$	4	4	−15	
$\frac{3}{2}$	4	9		
$-\frac{3}{4}$	4	−3		
4	4			

(7)

(8) $2x^4 - 3x^3 - x^2 - 6x - 10 = 0$ $\quad \frac{1\ 2\ 5\ 10}{1, 2}$

	2	−3	−1	−6	−10
−1	2	−5	4	−2	−8
1	2	−1	−2	−8	−18
2	2	1	1	−4	−18
−2	2	−7	13		

(11) $16x^5 - 16x^4 - 72x^3 + 72x^2 + 81x - 81 = 0$

	16	−16	−72	72	81	−81
1	16	0	−72	0	81	0

$16x^4 - 72x^2 + 81$ Let $a = x^2$

$16a^2 - 72a + 81$

$(4a - 9)(4a - 9) = (4a - 9)^2 \Rightarrow$

$4x^2 - 9 \quad 4x^2 - 9$

$(2x-3)(2x+3)(2x-3)(2x+3)$

$x \in \{1, -\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\}$

(15) $x^4 - 10x^2 + 8 = 0$

		0	−10	0	+8	
−4	1	−4	−6	−24	+	] 1
−3	1	−3	−1	3	−	
−2	1	−2	−6	12	−	
−1	1	−1	−9	9	−	] 2
0					+	
1	1	1	−9	0	−	] 3
2	1	2	−6	−12	−	
3	1	3	−1	−3	−	] 4
4	1	4	6	24	+	

(16) $x^4 - 10x^2 + 10 = 0$

	1	0	−10	0	10	
−5	1	−5	15	−75	+	
−4	1	−4	6	−24	+	
−3	1	−3	−1	3	+	] 1
−2	1	−2	−6	12	−	
−1	1	−1	−9	9	+	] 2
0					+	
1	1	1	−9	−9	+	] 3
2	1	2	−6	−12	−	
3	1	3	−1	−3	+	] 4

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21 $(x-2\sqrt{3})(x+2\sqrt{3}) = x^2 - 12$

$\frac{p}{q} \in \{\pm 1, 2, 3, 4, 6, 12\}$

	1	0	−12
1	1	1	−11
−1	1	−1	−11
2	1	2	−8
−2	1	−2	−8
±3	1	3	−3
±4	1	4	4
±6	1	6	24
±12	1	12	132

144

Since all rational choices have been used as solutions & we know $2\sqrt{3}$ is solution we know $2\sqrt{3}$ is irrational

Oral ex

(1) There may be several Roots between two consecutive intgrs.

(2) all Sgns are >0 ∴ all values >0 & cannot sum to 0

(3) If Neg roots the odd Powers are Neg & would yield values >0.

(4) all values would be <0 ∴ Not sum to 0.

(5) 1 positive root
$-x^3+2x^2-3x-4$ 2 or 0 Neg roots

(6) 1 >0 root
$-x^3-2x^2+3x-4$ 2 or 0 Neg roots

(7) 6, 4, 2, 0 Pos Roots
$x^6+4x^5+2x^4+7x^3+2x^2+x+3$ No Neg roots.

(8) No Pos Roots 0 upper bound.
∴ all 6 must be Negtive (0 is a root).

(9)
(10)
(11) } Thm 12 Pg 100

① $2x^3-3x^2-6x+6=0$

3 is upper bound - all >0

3	2	-3	-6	6	
	2	3	0	6	+
2	2	1	-4	-2	-
1	2	-1	-7	-1	-
0	2	-3	-6	6	+
-1	2	-5	-1	7	+
-2	2	-7	8	-10	-

② $16x^3+24x^2-25x-38=0$ $\quad -16x^3+24x^2+25x-38$

	16	24	-25	-38	
1	16	40	15	-23	
2	16	56	87	136	Upper Bound, between 1, 2
0				-38	
-1	16	8	-33	-5	
-2	16	-8	-9	-20	
-3	16	-24	47	-179	Lower bound alternating

136, -23, -38, -5, -20

∴ roots must be between

③ $7x^3-15x^2-16x+10=0$ $\quad -7x^3-15x^2+16x+10$

2 or 0 positive, 1 Neg root.

④ $8x^3+9x^2-7x-8$

1 Positive & 2 or 0 Neg roots.

⑤ $x^5-x^3+x+3=0$ $\quad -5x^5+x^3-x+3$

2 or 0 positive roots, 3 or 1 Neg roots.

⑥ $x^4+3x^2-2=0$

1 Pos Root.

1 Neg root.

(7) $x^5 - x^2 + 2x - 1 = 0$ $\quad -x^5 - x^2 - 2x - 1$

3 or 1 Pos Root. No Neg roots.

(8) $x^6 + 2x^3 - x^2 + 4 = 0$ $\quad x^6 - 2x^3 - x^2 + 4$

2 or 0 pos Roots. 2 or 0 Neg Roots.

9 (3) $7x^3 - 15x^2 - 16x + 10 = 0$

2 pos Roots or 0.

	7	−15	−16	10	
3	7	6	2	16	Upper bound.
−1	7	−22	6	4	
−2	7	−29	42	−74	Lower bound.

10 (4) $8x^3 + 9x^2 - 7x - 8 = 0$

	8	9	−7	−8	
2	8	25	43		
1	8	17	10		Upper Bound.
−1	8				
−2	8	−7	7		Lower Bound.

11 (5) $x^5 - x^3 + x + 3 = 0$

	1	−1	1	3		
2	1	0	1	4		Upper bound.
−1	1	−2	3	0		
−2	1	−3	7	−11		Lower bound.

12 (6) $x^4 + 3x^2 - 2 = 0$

	1	0	3	0	−2	
1	1	1	4	4	2	Upper.
−1	1	−1	4	−4	2	Lower bound.

13(7) $x^5 - x^2 + 2x - 1 = 0$

	1	0	0	−1	2	−1	
1	1	1	1	0	2	1	Upper
−1	1	−1	1	−2	4	−5	Corner,
−2	1	−2	4	−9	20	−41	

14(8) $x^6 + 2x^3 - x^2 + 4 = 0$

	1	0	0	2	−1	0	4	
1	1	1	1	3	2	2	6	Upper.
−1	1	−1	1	1				
−2	1	−2	4	−6	11	−22	48	Corner.

15 $x^5 + x^2 + 4 = 0$ No Positive roots.
$-x^5 + x^2 + 4 = 0$ + 1 Neg root
1 real root.

16 $4x^5 - 3x^4 - x^2 - 2 = 0$
1 Positive,
0 Neg roots,
1 real root.

17 $x^5 + 2x^4 + x^3 + 2x^2 + 3x + 6$ No Pos
$-x^5 + 2x^2 - x^3 + 2x^2 - 3x + 6$
5, 3, 1 Neg roots.
1 2 1 2 3 6

4 −3 +0 −2
4 −3 0 −1 0 −2

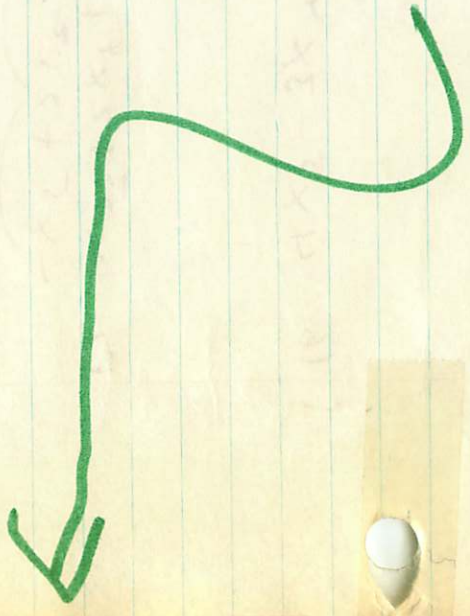
−1 4 −7 17 −8 8 −10

relatively prime

$\frac{4}{9} \cdot 2$

$\frac{4 \cdot 8}{9} = \frac{32}{9}$

still relatively prime.



1) $\frac{P}{Q}$ is a rational root (and relatively prime) of : $F(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots a_n = 0$

where a_0, a_1, $\cdots$, $a_n \varepsilon$ J then $P - MQ$ must divide $F(M)$

3) Simplified version : $P - Q$ must divide $F(1)$ and $P + Q$ must divide $F(-1)$.

PROOF :

4) Expand $F(x)$ in terms of $(x - M)$ to get : See example #4 pg 83. 2 things to notice: a_0 does not change. $F(M) = k_n$

5) $a_0(x - M)^n + k_1(x - M)^{n-1} + k_2(x - M)^{n-2} + \cdots + k_{n-1}(x - M) + k_n = 0$ where $k_i \varepsilon$ J

6) Since $x - M = 0$ when $x = M$, $F(M) = k_n$, Find $F\left(\frac{P}{Q}\right)$

7) $a_0\left(\frac{P}{Q} - M\right)^n + k_1\left(\frac{P}{Q} - M\right)^{n-1} + k_2\left(\frac{P}{Q} - M\right)^{n-2} + \cdots + k_{n-1}\left(\frac{P}{Q} - M\right) + k_n = 0$

8) $a_0\left(\frac{P - MQ}{Q}\right)^n + k_1\left(\frac{P - MQ}{Q}\right)^{n-1} + k_2\left(\frac{P - MQ}{Q}\right)^{n-2} + \cdots + k_{n-1}\left(\frac{P - MQ}{Q}\right) + k_n = 0$

9) $\frac{1}{Q^n}a_0(P - MQ)^n + \frac{1}{Q^{n-1}}k_1(P - MQ)^{n-1} + \frac{1}{Q^{n-2}}k_2(P - MQ)^{n-2} + \cdots + \frac{1}{Q}k_{n-1}(P - MQ) + k_n = 0$

10) $a_0(P - MQ)^n + Qk_1(P - MQ)^{n-1} + Q^2k_2(P - MQ)^{n-2} + \cdots + Q^{n-1}k_{n-1}(P - MQ) = -k_nQ^n$

11) $a_0(P - MQ)^{n-1} + Qk_1(P - MQ)^{n-2} + Q^2k_2(P - MQ)^{n-3} + \cdots + Q^{n-1}k_{n-1} = \frac{-k_nQ^n}{(P - MQ)}$

NOTE: The **left side** is the sum of integers, therefore the right side **MUST be an integer!**

13) $\frac{(P - MQ)}{Q} = \frac{P}{Q} - M$. Since $\frac{P}{Q}$ is relatively prime, so is $\frac{P}{Q} - M$ and $\frac{(P - MQ)}{Q}$.

14) AND so is $\frac{Q}{(P - MQ)}$ (since the reciprocal does not change common factors) and so is $\frac{Q^n}{(P - MQ)}$.

15) Conclusion : $P - MQ$ divides $-k_nQ^n$. Since $P - MQ$ does NOT divide Q^n, it MUST divide $-k_n$

16) Then $P - MQ$ must divide $F(M)$. For simplified use, let $M = \pm 1$ to yield :

17) P - Q divides $F(1)$ AND $P + Q$ divides $F(-1)$

Solve for all roots over the complex numbers. Show the RRTS at work.

(1) $2x^5 - 11x^4 + 14x^3 - 2x^2 + 12x + 9 = 0$
(2) $12x^5 + 8x^4 + 11x^3 + 7x^2 - x - 1 = 0$
(3) $2x^3 - 3x^2 - x + 12 = 0$
(4) $2x^3 - 7x^2 - 21x + 54 = 0$
(5) $4x^4 - 12x^3 + x^2 + 14x - 3 = 0$
(6) $2x^5 - 11x^4 + 4x^3 + 53x^2 - 48x - 36 = 0$
(7) $12x^4 - 56x^3 + 59x^2 + 9x - 18 = 0$
(8) $3x^4 + 46x^3 - 30x^2 + 33x + 16 = 0$
(9) $16x^4 - 72x^2 + 81 = 0$
(10) $10x^4 - 27x^3 - 2x^2 + 44x - 24 = 0$
(11) $6x^4 - 35x^3 + 64x^2 - 15x - 50 = 0$
(12) $6x^4 + 13x^3 - 7x^2 - 26x - 10 = 0$
(13) $270x^2 - 453x + 287x - 40 = 0$
(14) $10x^4 - 130x^2 + 360 = 0$
(15) $24x^3 + 50x^2 - 29x - 10 = 0$
(16) $192x^4 + 40x^3 - 66x^2 + 9x = 0$
(17) $75x^3 - 10x^2 - 96x - 32 = 0$
(18) $12x^3 + 112x^2 + 343(x + 1) = 0$
(19) $128x^3 + 48x^2 - 200x - 75 = 0$
(20) $36x^4 - 24x^3 - 95x^2 + 75x = 0$

ERRTS (simplified) An Example:

2 or 0 Real Roots > 0

$24x^2+42x^3-77x^2-56x+40=0$

2 or 0 Real < 0 Roots

Theorem: If $\frac{P}{Q}$ is a rational root (and relatively prime) of $F(x)$ then $(P-Q)$ divides $F(1)$ and $(P+Q)$ divides $F(-1)$.

Given: $F(x) = 24x^4 - 42x^3 - 77x^2 + 56x + 60 \overset{set}{=} 0$ $\quad \pm\left\{\frac{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 10,\ 12,\ 15,\ 20,\ 30,\ 60}{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24}\right\}$

	24	−42	−77	56	60	
−1]	24	−66	−11	67	−7	$= F(-1)$
+1]	24	−18	−95	−39	21	$= F(1)$

$\pm\left\{\frac{5}{4}, \frac{5}{6}, \frac{5}{8}, \frac{5}{12}, \frac{5}{24}, \frac{6}{1}, \frac{10}{1}, \frac{12}{1}, \frac{15}{1}, \frac{15}{2}, \frac{15}{4}, \frac{15}{8}, \frac{20}{1}, \frac{30}{1}, \frac{60}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{12}, \frac{1}{24}, \frac{2}{1}, \frac{2}{3}, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{4}{1}, \frac{4}{3}, \frac{5}{1}, \frac{5}{2}, \frac{5}{3}\right\}$

Step #1 of ERRTS: ADD and/or SUBTRACT denominator to/from numerator.

If either divides $F(-1)$, then it is a possible root. See example:

$\frac{1}{2} \rightarrow \frac{1+2=3 \text{ does NOT divide } -7}{1-2=-1 \text{ does divide } -7}$ $\quad -\frac{1}{2}$ is a possible root

because the result was found by *subtraction*

$\frac{1}{3} \rightarrow \frac{1+3=4 \text{ NO}}{1-3=2 \text{ NO}}$ $\frac{3}{4} \rightarrow \frac{3+4=7 \text{ YES}}{3-4=-1 \text{ YES}}$ $\pm\frac{3}{4}$ etc.

make a list of all of our possibilities:

$\left\{\frac{1}{6}, \frac{3}{4}, \frac{4}{3}, \frac{5}{2}, \frac{-1}{2}, \frac{-1}{8}, \frac{-2}{1}, \frac{-2}{3}, \frac{-3}{2}, \frac{-3}{4}, \frac{-4}{3}, \frac{-5}{4}, \frac{-5}{6}, \frac{-5}{12}, \frac{6}{1}, \frac{-10}{3}, \frac{-15}{8}\right\}$

Step # 2 of ERRTS: SUBTRACT only the denominator from the numerator.

If the result divides $F(1) = 21$ then it is a possible root.

$\frac{1}{6} \rightarrow 1-6=-5$ NO $\quad \frac{3}{4} \rightarrow 3-4=-1$ YES

$\frac{-1}{2} \rightarrow -1-2=-3$ YES $\quad \frac{5}{2} \rightarrow 5-2=3$ YES etc.

$\rightarrow \left\{\frac{3}{4}, \frac{4}{3}, \frac{5}{2}, \frac{-1}{2}, \frac{-3}{4}, \frac{-4}{3}, \frac{-2}{1}\right\}$

	24	−42	−77	56	60	
$\frac{3}{4}$]	24	−24	−95	X		
$\frac{4}{3}$]	24	−10	X			
$\frac{5}{2}$]	24	18	−32	−24	0	**Root** other pos root must be irrational
	12	9	−16	−12		
$\frac{-1}{2}$]	12	3	X			
$\frac{-3}{4}$]	12	0	−16	0		**Root** Now I have a quadradic equation

$12x^2 - 16 = 0$

$3x^2 - 4 = 0$

$3x^2 = 4$

$x^2 = \frac{4}{3}$

$x = \pm\frac{2\sqrt{3}}{3}$

The roots of $F(x)$ are:

$\left\{\frac{5}{2}, -\frac{3}{4}, \pm\frac{2\sqrt{3}}{3}\right\}$

and the factors of $F(x)$ are:

$(2x-5)(4x+3)\left(3x^2-4\right)$

RRTS
1991

(1) $2x^5 - 11x^4 + 14x^3 - 2x^2 + 12x + 9 = 0$ $\left\{-\frac{1}{2}, 3, 3, \pm i\right\}$

(2) $12x^5 + 8x^4 + 11x^3 + 7x^2 - x - 1 = 0$ $\left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, \pm i\right\}$

(3) $2x^3 - 3x^2 - x + 12 = 0$ $\left\{-\frac{3}{2}, \frac{3 \pm i\sqrt{7}}{2}\right\}$

(4) $2x^3 - 7x^2 - 21x + 54 = 0$ $\left\{2, -3, \frac{9}{2}\right\}$

(5) $4x^4 - 12x^3 + x^2 + 14x - 3 = 0$ $\left\{-1, \frac{3}{2}, \frac{5 \pm \sqrt{17}}{4}\right\}$

(6) $2x^5 - 11x^4 + 4x^3 + 53x^2 - 48x - 36 = 0$ $\left\{-2, 2, 3, 3, -\frac{1}{2}\right\}$

(7) $12x^4 - 56x^3 + 59x^2 + 9x - 18 = 0$ $\left\{-\frac{1}{2}, \frac{3}{2}, 3, \frac{2}{3}\right\}$

(8) $3x^4 + 46x^3 - 30x^2 + 33x + 16 = 0$ $\left\{-\frac{1}{3}, -16, \frac{1 \pm \sqrt{3}}{2}\right\}$

(9) $16x^4 - 72x^2 + 81 = 0$ $\left\{\pm\frac{3}{2}\right.$ both multiplicity 2$\left.\right\}$

(10) $10x^4 - 27x^3 - 2x^2 + 44x - 24 = 0$ $\left\{2, \frac{6}{5}, \frac{1 \pm \sqrt{17}}{4}\right\}$

(11) $6x^4 - 35x + 64x^2 - 15x - 50 = 0$ $\left\{-\frac{2}{3}, \frac{5}{2}, 2 \pm i\right\}$

(12) $6x^4 + 13x^3 - 7x^2 - 26x + 40 = 0$ $\left\{-\frac{1}{2}, -\frac{5}{3}, \pm\sqrt{2}\right\}$

(13) $270x^2 - 453x + 287x - 40 = 0$ $\left\{\frac{4}{51}, \frac{-5}{27}\right\}$

$270x^2 - 166x - 40 = 0$

$135x^2 - 83x - 20 = 0$

RRTS
#2
1991

(14) $10x^4 - 130x^2 + 360 = 0$ $\{\pm 2, \pm 3\}$

$x^4 - 13x^2 + 36 = 0$

$(x^2 - 9)(x^2 - 4)$

$x = \pm 3 \quad x = \pm 2$

(15) $24x^3 + 50x^2 - 29x - 10 = 0$ $\left\{\frac{2}{3}, -\frac{1}{4}, -\frac{5}{2}\right\}$

(16) $192x^4 + 40x^3 - 66x^2 + 9x = 0$

(17) $75x^3 - 10x^2 - 96x - 32 = 0$

(18) $12x^3 + 112x^2 + 343x + 343 = 0$

(19) $128x^3 + 48x^2 - 200x - 75 = 0$

(20) $36x^4 - 24x^3 - 95x^2 + 75 = 0$

Honors Analysis Chapt 2 Chapt 2

Solve for x

(1) $x^3+4x^2+8x+5=0$

(2) $2x^5-11x^4+14x^3-2x^2+12x+9=0$

(3) $x^3+\frac{1}{5}x^2-x-\frac{1}{5}=0$

(4) $3x^4-37x^3+147x-207x+54=0$

(5) $16x^4+32x^3-32x-16=0$

(6) $12x^4+x^3-54x^2-4x+24=0$

(7) $12x^5+8x^4+11x^3+7x^2-x-1=0$

(8) $3x^3-10x^2+7x+10=0$

(9) $x^5+6x^4+10x^3-4x^2-24x-16=0$

(10) $24x^5+14x^4-105x^3-62x^2+44x+24=0$

(11) $x^4+3x^3-9x-27=0$

(12) $8x^4-5x^3-115x^2+150x+72=0$

(13) $2x^3-3x^2-4x+1=0$

(14) $16x^4-18x^3-41x^2+20x+12=0$

(15) $48x^6+4x^5-224x^4-19x^3+150x^2+4x-24$

RRT S Ditto

① $x^3+4x^2+8x+5=0$

a) No positive Roots.

b) 3 or 1 Negative Roots.

c) $\pm\{1, 5\}$

Descartes Rule of Signs.

$x \in \{-1, \frac{-3 \pm i\sqrt{11}}{2}\}$

	1	4	8	5	
-1	1	3	5	0	Root

$x^2+3x+5=0$

$x = \frac{-3 \pm \sqrt{9-20}}{2} = \frac{-3 \pm i\sqrt{11}}{2}$

② $2x^5-11x^4+14x^3-2x^2+12x+9=0$

a) 4, 2 or 0 + Roots.

b) 1 Negative Root.

c) $\pm\{\frac{1\ 3\ 9}{1\ 2} \Rightarrow \pm\{1, \frac{1}{2}, \frac{3}{1}, \frac{3}{2}, \frac{9}{1}, \frac{9}{2}\}$

Result using RRT,S

	2	-11	14	-2	12	9	ALT signs ∴ Lower bound.
-1	2	-13	27	-29	41	-32	$\{-\frac{1}{2}, \frac{\pm 3}{1}, -\frac{3}{2}, -\frac{9}{1}\}$
+1	2	-9	5	3	15	24	$\{-\frac{1}{2}, \pm 3\}$
3	2	-5	-1	-5	-3	0	Root. Depressed Equation.
3	2	1	+2	1	0		Root, ENDS IN 1 which is not divisible by 3 ∴ we don't try +3 again.
$-\frac{1}{2}$	2	0	2	0			Root.

$2x^2+2=0$

$x^2=-1$

$x=\pm i$

Lower bound.

Not upper bound since not all >0.

STILL NOT

$x \in \{3, 3, -\frac{1}{2}, \pm i\}$.

Upper Bound.

These items is quadratic — factor or use Quad Formula.

$448 = 4 \cdot 112$
$8 \cdot 56$
$16 \cdot 28$
$32 \cdot 14$
$64 \cdot 7$

RRTS
D1 T6

④ $3x^4 - 37x^3 + 147x^2 - 207x + 54 = 0$

+ + + + +

a) 4, 2, 0 + roots. No neg Roots.

$\pm\{1\ 2\ 3\ 6\ 9\ 18\ 27\ 54\}$ / $\{1\ 3\}$

Delete: $\pm 2, -6, \pm 9, \pm 18, \pm 27, \pm 54$

new list $\{\pm 2, \pm\frac{2}{3}, \pm 3, +6, -9\}$

	3	−37	147	−207	54	
−1 ⌋	3	−40	187	−394	448	lower b and delete
+1 ⌋	3	−34	113	−94	−40	No help in shortening list,
+3 ⌋	3	−28	63	−18	0	
3 ⌋	3	−19	6	0		

$3x^2 - 19x + 6 = 0$

$(3x - 1)(x - 6)$

$x \in \{\frac{1}{3}, 6, 3, 3\}$

$\frac{19 \pm \sqrt{361 - 72}}{6}$

$\frac{19 \pm 17}{6}$ $\quad \frac{2}{6}\ \frac{1}{3}$

⑤ $16x^4 + 32x^3 - 32x - 16 = 0 \Rightarrow x^4 + 2x^3 - 2x - 1 = 0$

1 + Root

3, 1 − Roots $\{\pm 1\}$

only 1 + Root So

$x \in \{1, -1, -1, -1\}$

+ − + −

	1	2	0	−2	−1
1 ⌋	1	3	3	1	0 ← 43
−1 ⌋	1	+2	1	0	

$x^2 + 2x + 1$

6 $12x^5+8x^4+11x^3+7x^2-x-1=0$

+ − + + −

1 + Root

4, 2, 0 − Roots.

$\pm\left\{\frac{1}{1,2,3,4,6,12}\right.$

Delete: $\frac{1}{2}\ \pm\frac{1}{4}\ \pm\frac{1}{6}\ \pm\frac{1}{12}$

New list. $-\frac{1}{2}, \pm\frac{1}{3}$

	12	8	11	7	−1	−1	
−1	12	−4	15	−8	7	−8	
1	12	20	31	38	37	36	no help.
$-\frac{1}{2}$	12	2	10	2	−2	0	
$-\frac{1}{2}$	6	1	5	1	−1		←
	6	−2	6	−2	0		
	3	−1	3	−1			←
$\frac{1}{3}$	3	0	3	0			

$x\in\left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, \pm i\right\}$

$3x^2+3=0$

$x=\pm i$.

(7) $3x^3-10x^2+7x+10=0$

2 or 0 + Roots

1 − Root.

$\pm\ \frac{1,2,5,10}{1,3}$

Delete $+2, \pm 5, \pm 10, +\frac{5}{3}, \pm\frac{10}{3}$.

New list $-2, \pm\frac{2}{3}, -\frac{5}{3}$

	3	−10	7	10
−1	3	−13	20	−10
1	3	−7	0	10 no help again.
−2	3	−14		
$-\frac{2}{3}$	3	−12	15	0

$1\quad -4\quad 5$

$x^2-4x+5=0$

$\frac{4\pm\sqrt{16-20}}{2}\quad \frac{4\pm 2i}{2}=2\pm i$

$x\in\left\{-\frac{2}{3}, 2\pm i\right\}$

⑧ $x^5+6x^4+10x^3-4x^2-24x-16=0$ 1 + Root
− + − − + − 4, 2, 0 − Roots.
RRTS DITU

$\pm\{1, 2, 4, 8, 16\}$

	1	6	10	−4	−24	−16	Delete, ±4, ±8, ±16
−1	1	5	5	−9	−15	−1	new list ±2
1	1	7	17	13	−11	−27	No help - as usual,
−2	1	8	26	48	+	+	u 13
−2	1	4	2	−8	−8	0	Root
−2	1	2	−2	−4	0		Root
−2	1	0	−2	0			Root

$x^2-2=0$

$x=\pm\sqrt{2}$ $\qquad x\in\{-2,-2,-2,\pm\sqrt{2}\}$

⑨ $x^4+3x^3-9x-27=0$ 1 + root 2 or 0 − Roots.
+ − + − must be some imag Roots,

$\pm$ 1 3 9 27

1

Delete −9 ±27

$x^3-9=0$

$x^3=9$

$x=\sqrt[3]{9}$

	1	3	0	−9	−27	
−1	1	2	−2	−7	−20	
1	1	4	4	−5	−32	No help.
3	1	6	18	45	108	uB
−3	1	0	0	−9	0	Root
$\sqrt[3]{9}$	1	$\sqrt[3]{9}$	$\sqrt[3]{9^2}$	0		

$x^2+\sqrt[3]{9}x+\sqrt[3]{81}=0$ Let $\sqrt[3]{9}=z$

$x^2-zx+z^2=0$

$x=\dfrac{-z\pm\sqrt{z^2-4z^2}}{2}=\dfrac{-z\pm zi\sqrt{3}}{2}$

$\sqrt[3]{9}=3^{\frac{2}{3}}$

$3^{\frac{2}{3}}\cdot 3^{\frac{1}{2}}=3^{\frac{7}{6}}$

$x\in\left\{-3,\ \sqrt[3]{9},\ \dfrac{-\sqrt[3]{9}\pm\sqrt[3]{9}\,i\sqrt{3}}{2}\right\}$

$\dfrac{-\sqrt[3]{9}\pm 3i\sqrt[6]{3}}{2}$

(10) $8x^4 - 5x^3 - 115x^2 + 150x + 72 = 0$

2 or 0 + Roots
2 or 0 − Roots

$+ \quad + \quad - \quad + \quad +$

$\pm \left\{ \frac{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72}{1, 2, 4, 8} \right\}$

$2^3, 3, 5$
$11 \cdot 2 \cdot 5$

Delete $\pm\frac{1}{8}, \pm\frac{3}{4}, \pm\frac{3}{8}, \pm\frac{6}{1}, \pm 8, \pm\frac{9}{2}, \frac{9}{4}, \frac{9}{8}, \pm 12, \pm 18, \pm 24, \pm 36, \pm 72$

	8	−5	−115	+150	72	
−1 ⌋	8	−13	−102	−48	120	New list $\frac{1}{2}, \frac{-1}{2}, \frac{1}{4}, \frac{-1}{4}, 2, -2, 3, -3, \frac{3}{2}, \frac{-3}{2}$
1 ⌋	8	3	−112	+38	120	$\frac{-3}{4}, \frac{-3}{8}, 4, -4, -6, 9, -9, \frac{-9}{4}, \frac{-9}{8}$
						No help again.
we try for upper bound 2 ⌋	8	11	−93	−36	0	Root. But not upper bound.
−1 ⌋	8	3	−96	60		Trying to reduce New list. Delete −9.
1 ⌋	8	19	−74	−110		New list $\frac{1}{2}, \frac{-1}{4}, 2, 3, \frac{3}{2}, \frac{-3}{8}, -4, -9$
3 ⌋	8	35	12	0		Root.

$8x^2 + 35x + 12 = 0$

$(8x + 3)(x + 4)$

$x \in \left\{ 2, 3, -4, -\frac{3}{8} \right\}$

$-\frac{3}{8}$

(11) $2x^3 - 3x^2 - 4x + 1 = 0$

$\pm\left\{ 1, \frac{1}{2} \right\}$

2, or 0 + roots
1 − Root.

	2	−3	−4	1
1 ⌋	2	−1	−5	−4
+1 ⌋	2	−5	1	0

$2x^2 - 5x + 1 = 0$

$\frac{5 \pm \sqrt{25 - 8}}{4}$

$x = \frac{5 \pm \sqrt{17}}{4}$

$x \in \left\{ -1, \pm \frac{5 \pm \sqrt{17}}{4} \right\}$

RRTS Ditto

+ + − − +

(12) $16x^4 - 18x^3 - 41x^2 + 20x + 12 = 0$ 2,0 + Roots. 2,0 − Roots.

$\pm\left\{\frac{1,2,3,4,6,12}{1,2,4,8,16}\right\}$

	16	−18	−41	20	12	
−1	16	−34	−7	27	−15	
1	16	−2	−43	−23	−11	
$\frac{1}{2}$	16	−10	−46	−3		
→ 2	16	14	−13	−6	0	Root.
$\frac{3}{2}$	16	32	35			
$-\frac{3}{8}$	16	−8	−16	0		Root

New list $\left\{\pm\frac{1}{2}, \pm\frac{1}{4}, -\frac{1}{16}, \pm 2, \pm\frac{3}{2}, -\frac{3}{4}, -\frac{3}{8}, \pm 4, -6,\right\}$

New list $\left\{\pm\frac{1}{2}, +2, +\frac{3}{2}, -\frac{3}{8},\right\}$

2 1 −16

$2x^2 + x - 2 = 0$

$x = \frac{-1 \pm \sqrt{1+16}}{4}$

$x = \frac{-1 \pm \sqrt{17}}{4}$

$x \in \left\{2, -\frac{3}{8}, \frac{-1 \pm \sqrt{17}}{4}\right\}$

1 2 3 4 5 6 7 8

$\frac{2}{5}$ $\frac{1}{6}$ $\frac{3}{5}$ $\frac{5}{6}$ $\frac{1}{4}$ $\frac{3}{4}$

$(5x-2)(6x-1)$

$90x^5+459x^4+257x^3-512x^2+172x-16$

$30x^2-17x+2$

$x+4$

$30x^3-17x^2+2x$

$120x^2-68x+8$

$30x^3+103x^2-66x+8$ $\{\frac{2}{5}\ \frac{1}{6}\ -4\}$

$x+2$

(1) $30x^4+103x^3-66x^2+8x$

$60x^3+206x^2-132x+16$

$30x^4+163x^3+140x^2-124x+16$

$\{\frac{2}{5}\ \frac{1}{6}\ -4\ -2\}$

$\{\frac{2}{5}\ 1\ 6\}$

$3x-1$

	30	163	140	−124	16
		103	−66	8	0
−2	30	−17	2	0	
−4	30	−12	0		
$\frac{1}{6}$	30	0			
$\frac{2}{5}$					

$90x^5+489x^4+420x^3-372x+48$

$-30x^4-163-140+124-16$

$90x^5+459x^4+257x^3-512x+172-16$

	90	459	+257	−512	172	−16
		30	163	140	124	
$\frac{1}{3}$	90	489	420	−372	48	0

~~$90x^5+489x^4+420x^3-372x+48$~~

13 $(5x-2)(6x-5)(9x-4)$

$30x^2 - 37x + 10$

$9x - 4$

$270x^2 - 333x + 90x$

$-120x + 148x - 40$

$270x^2 - 453x + 238x - 40$

$270 \quad -453 \quad +238 \quad -40$

$\frac{5}{6}$ | $270 \quad -228 \quad +48 \quad 0$

$135 \quad -114 \quad +24$

$45 \quad -38 \quad +8$

$\frac{2}{5}$ | $45 \quad -30 \quad +20 \quad 0$

$9 \quad -4$

$448 = 4 \cdot 112$
$8 \cdot 56$
$16 \cdot 28$
$32 \cdot 14$
$64 \cdot 7$

RRTS
D1 | 18

④ $3x^4 - 37x^3 + 147x^2 - 207x + 54 = 0$

$+ \quad + \quad + \quad +$

a) 4, 2, 0 + Root. No neg Roots.

$\pm \{1\ 2\ 3\ 6\ 9\ 18\ 27\ 54\}$
$\{1\ 3\}$

Delete: $+2, -6, +9, \pm 18, \pm 27, \pm 54$
new list $\{-2, \pm\frac{2}{3}, \pm 3, +6, -9\}$

	3	−37	147	−207	54	
−1	3	−40	187	−394	448	Lower bound and delete
+1	3	−34	113	−94	−40	No help in shortening list,
+3	3	−28	63	−18	0	
3	3	−19	6	0		

$3x^2 + 19x + 6 = 0$
$(3x + 1)(x + 6)$

$x \in \{-\frac{1}{3}, -6, 3, 3\}$

$\frac{19 \pm \sqrt{361 - 72}}{6}$

$\frac{19 \pm 17}{6} \quad \frac{2}{6} \quad \frac{1}{3}$

⑤ $16x^4 + 32x^3 - 32x - 16 = 0 \Rightarrow x^4 + 2x^3 - 2x - 1 = 0$

$+ \quad - \quad + \quad -$

1 + Root
3, 1 − Roots $\{\pm 1\}$
only 1 + Root So

	1	2	0	−2	−1	
1	1	3	3	1	0	4 3
−1	1	+2	1	0		

$x^2 + 2x + 1$

$x \in \{1, -1, -1, -1\}$

6 $12x^5+8x^4+11x^3+7x^2-x-1=0$

VARS DITO

$+ \quad - \quad + \quad + \quad -$

1 + Root

4, 2, 0 − Roots.

$\pm\left\{\frac{1}{1,2,3,4,6,12}\right.$

	12	8	11	7	−1	−1	Delete: $\frac{1}{2}\ \pm\frac{1}{4}\ \pm\frac{1}{6}\ \pm\frac{1}{12}$
-1	12	−4	15	−8	7	−8	New list. $-\frac{1}{2}, \pm\frac{1}{3}$
1	12	20	31	38	37	36	no help.
$-\frac{1}{2}$	12	2	10	2	−2	0	
$-\frac{1}{2}$	6	1	5	1	−1		←
	6	−2	6	−2	0		
	3	−1	3	−1			←
$\frac{1}{3}$	3	0	3	0			

$x\in\left\{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, \pm i\right\}$

$3x^2+3=0$

$x=\pm i$

(7) $3x^3-10x^2+7x+10=0$

2 or 0 + Roots

1 − Root.

$\pm\ \frac{1,2,5,10}{1,3}$

Delete $+2, \pm5, \pm10, +\frac{5}{3}, \pm\frac{10}{3}$

New list $-2, \pm\frac{2}{3}, -\frac{5}{3}$

	3	−10	7	10
-1	3	−13	20	−10
1	3	−7	0	10 no help again.
-2	3	−14		
$-\frac{2}{3}$	3	−12	15	0
	1	−4	5	

$x^2-4x+5=0$

$\frac{4\pm\sqrt{16-20}}{2}$ $\quad \frac{4\pm 2i}{2}=2\pm i$

$x\in\left\{-\frac{2}{3}, 2\pm i\right\}$

⑧ $x^5+6x^4+10x^3-4x^2-24x-16=0$ 1 + Root
− + − − + − 4, 2, 0 − Roots.
RRTS DITA

$\pm\{1,2,4,8,16\}$

	1	6	10	−4	−24	−16	Delete, ±4, ±8, ±16
−1	1	5	5	−9	−15	−1	new list ±2
1	1	7	17	13	−11	−27	No help - as usual,
−2	1	8	26	48	+	+	u 3
−2	1	4	2	−8	−8	0	Root
−2	1	2	−2	−4	0		Root
−2	1	0	−2	0			Root

$x^2-2=0$

$x=\pm\sqrt{2}$ $\qquad x\in\{-2,-2,-2,\pm\sqrt{2}\}$

⑨ $x^4+3x^3-9x-27=0$ 1 + root 2 or 0 − Roots.
+ − + − must be some imag Roots,

$\pm$ 1 3 9 27

1

Delete −9 ±27

$x^3-9=0$

$x^3=9$

$x=\sqrt[3]{9}$

	1	3	0	−9	−27	
−1	1	2	−2	−7	−20	
1	1	4	4	−5	−32	No help.
3	1	6	18	45	108	uB
−3	1	0	0	−9	0	Root
$\sqrt[3]{9}$	1	$\sqrt[3]{9}$	$\sqrt[3]{9^2}$	0		

$x^2+\sqrt[3]{9}x+\sqrt[3]{81}=0$ Let $\sqrt[3]{9}=z$

$x^2-zx+z^2=0$

$$x=\frac{-z\pm\sqrt{z^2-4z^2}}{2}=\frac{-z\pm zi\sqrt{3}}{2}$$

$\sqrt[3]{9}=3^{\frac{2}{3}}$

$3^{\frac{2}{3}}\cdot 3^{\frac{1}{2}}=3^{\frac{7}{6}}$

$$x\in\left\{-3,\ \sqrt[3]{9},\ \frac{-\sqrt[3]{9}\pm\sqrt[3]{9}\,i\sqrt{3}}{2}\right\}$$

$$\frac{-\sqrt[3]{9}\pm 3i\sqrt[6]{3}}{2}$$

(10) $8x^4 - 5x^3 - 115x^2 + 150x + 72 = 0$

2 or 0 + Roots
2 or 0 − Roots

$\pm\left\{\frac{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72}{1, 2, 4, 8}\right\}$

$2^3 \cdot 3^2 \cdot 5$
$11 \cdot 2 \cdot 5$

Delete $\pm\frac{1}{8}\ \pm\frac{3}{4}\ \pm\frac{3}{8}, \pm\frac{6}{1}\ \pm\frac{9}{2}, \frac{9}{4}, \frac{9}{8}, \pm 12$
$\pm 18\ \pm 24, \pm 36\ \pm 72$

	8	−5	−115	+150	72	
−1	8	−13	−102	−48	120	New list $\frac{1}{2}\ \frac{-1}{2}\ \frac{1}{4}, \frac{-1}{4}, 2, -2, 3, -3, \frac{3}{2}, \frac{-3}{2}$
1	8	3	−112	+38	120	$\frac{-3}{4}\ \frac{-3}{8}\ 4, -4\ -6\ 9\ -9\ \frac{-9}{4}\ \frac{-9}{8}$
						No help again.
2	8	11	−93	−36	0	Root. But not upper bound.
−1	8	3	−96	60		Trying to reduce New list. Delete −9.
1	8	19	−74	−110		new list $\frac{1}{2}, \frac{-1}{4}, 2, 3, \pm\frac{3}{2}, \frac{-3}{8}, -4, -9$
3	8	35	12	0	Root.	

We try for upper bound

$8x^2 + 35x + 12 = 0$

$(8x + 3)(x + 4)$ $\quad x \in \left\{2, 3, -4, -\frac{3}{8}\right\}$ $\quad -\frac{3}{8}$

(11) $2x^3 - 3x^2 - 4x + 1 = 0$ $\quad \pm\left\{1, \frac{1}{2}\right\}$

2, or 0 + roots
1 − Root.

	2	−3	−4	1
1	2	−1	−5	−4
1	2	−5	1	0

$2x^2 - 5x + 1 = 0$

$\frac{5 \pm \sqrt{25 - 8}}{4}$

$x = \frac{5 \pm \sqrt{17}}{4}$

$x \in \left\{-1, \pm\frac{5 \pm \sqrt{17}}{4}\right\}$

RRTS Dille

(12) $16x^4 - 18x^3 - 41x^2 + 20x + 12 = 0$ $\quad + + - - +$ $\quad 2, 0$ + Roots. $\quad 2, 0$ − Roots.

$$\pm\left\{\frac{1, 2, 3, 4, 6, 12}{1, 2, 4, 8, 16}\right\}$$

	16	−18	−41	20	12	
-1	16	−34	−7	27	−15	
1	16	−2	−43	−23	−11	
$\frac{1}{2}$	16	−10	−46	−3		
2	16	14	−13	−6	0	Root.
$\frac{3}{2}$	16	32	35			
$-\frac{3}{8}$	16	−8	−16	0		Root.

New list $\left\{\pm\frac{1}{2}, \pm\frac{1}{4}, -\frac{1}{16}, \pm 2, \pm\frac{3}{2}, -\frac{3}{4}, -\frac{3}{8}, \pm 4, -6,\right\}$

New list $\left\{+\frac{1}{2}, +2, +\frac{3}{2}, -\frac{3}{8},\right\}$

$2 \quad 1 \quad -16$

$2x^2 + x - 2 = 0$

$$x = \frac{-1 \pm \sqrt{1+16}}{4}$$

$$x = \frac{-1 \pm \sqrt{17}}{4}$$

$$x \in \left\{2, -\frac{3}{8}, \frac{-1 \pm \sqrt{17}}{4}\right\}$$

1 2 3 4 5 6 7 8

$\frac{2}{5}$ $\frac{1}{6}$ $\frac{3}{5}$ $\frac{5}{6}$ $\frac{1}{4}$ $\frac{3}{4}$

$(5x-2)(6x-1)$

$30x^2 - 17x + 2$

$x + 4$

$30x^3 - 17x^2 + 2x$

$120x^2 - 68x + 8$

$30x^3 + 103x^2 - 66x + 8$ $\left\{\frac{2}{5}\ \frac{1}{6}\ -4\right\}$

$x + 2$

(5) $30x^4 + 103x^3 - 66x^2 + 8x$

$60x^3 + 206x^2 - 132x + 16$

$30x^4 + 163x^3 + 140x^2 - 124x + 16$

$\left\{\frac{2}{5}\ \frac{1}{6}\ -4\ 2\right\}$

$3x - 1$

	30	163	140	−124	16
		120 43		68	
−2	30	103	−66	8	0
−4	30	−17	2	0	
$\frac{1}{6}$	30	−12	0		
$\frac{2}{5}$	30	0			

$90x^5 + 489x^4 + 420x^3 - 372x + 48$

$-30x^4 - 163 - 140 + 124 - 16$

$90x^5 + 459x^4 + 257x^3 - 512x + 172 - 16$

	90	459	+257	−512	172	−16
		30	163	140	124	
$\frac{1}{3}$	90	489	420	−372	48	0

$90x^5 + 489x^4 + 420x^3 - 372x + 48$

13 $(5x-2)(6x-5)(9x-4)$

$30x^2 - 37x + 10$

$9x - 4$

$270x^2 - 333x + 90x$

$-120x + 148x - 40$

$270x^2 - 453x + 238x - 40$

$270 \quad -453 \quad +238 \quad -40$

$\frac{5}{6}$ | $270 \quad -228 \quad +48 \quad 0$

$135 \quad -114 \quad +24$

$45 \quad -38 \quad +8$

$\frac{2}{5}$ | $45 \quad -30 \quad 0$

$9 \quad -4$

$(5x-3)(4x+1)$

$20x^2 - 7x - 3$ ✓

$5x+2$

$100x^3 - 35x^2 - 15x$

$40x^2 - 14x - 6$

$100x^3 + 5x^2 - 29x - 6$ ✓

$2x+5$

$200x^4 + 10x^3 - 58x^2 - 12x$

$500x^3 + 25x^2 - 145x - 30$

(6)

$200x^4 + 510x^3 - 33x^2 - 157x - 30$ ✓

$x-3$

$-600x^4 - 1530x^3 + 99x^2 + 471x + 90$

$200x^5 + 510x^4 - 33x^3 - 157x^2 - 30x$

$200x^5 - 90x^4 - 1563x^3 - 158x^2 + 441x + 90$

	200	−90	−1563	−58	441	90	86
			1530		−471		172
3	200	510	−33	−157	−30	0	
		−80	−172	82			41
$-\frac{2}{5}$	200	430	−205	−75	0		82
							15
$-\frac{5}{2}$	40	86	−41	−15			
	40	−100	35	0			
		−14	−6				
$-\frac{1}{4}$	20	−7	−3				
	20	+12	0				
$\frac{3}{5}$	10	−6					
	5	−3					
		0					

$\frac{2}{5}$ $\frac{5}{6}$ $\frac{4}{9}$ -12 -20

(13)

$(5x-2)(6x-5)(9x-4)$

$30x^2-37x+10$

$9x-4$

$-120x^2+148x-40$

$270x^3-333x^2+90x$

$270x^3-453x^2+238x-40$

$\frac{3}{4}$ $-\frac{1}{8}$ $-\frac{10}{3}$ -3

$(4x-3)(8x+1)(3x+10)(x+3)$

(11)

$32x^2-20x-3$

$3x+10$

$96x^3-60x^2-9x$

$320x^2-200x-30$

$96x^3+260x^2-209x-30$

$x+3$

$96x^4+260x^3-209x^2-30x$

$288x^3+780x^2-627x-90$

$96x^4+448x^3+571x^2-657x-90$

571

780
209
571

$x+3=$

780
209
571

(1) $2x^5 - 11x^4 + 14x^3 - 2x^2 + 12x + 9 = 0$

$\frac{1, 3, 9}{1, 2}$

$-\frac{1}{2}, \frac{3}{1}$

	2	-11	14	-2	12	9
-1	2	-13	27	-29	41	-32
1	2	-9	5	3	15	24
$-\frac{1}{2}$	2	-12	20	-12	18	0
3	1	-6	10	-6	9	
3	1	-3	1	-3	0	
	1	0	1	0		

$x^2 + 1 = 0$

$x = \{-\frac{1}{2}, 3, 3, \pm i\}$

(2) $12x^5 + 8x^4 + 11x^3 + 7x^2 - x - 1 = 0$

$\frac{1}{1, 2, 3, 4, 6, 12}$

$-\frac{1}{2}, \frac{1}{3}, -\frac{1}{3},$

	12	8	11	7	-1	-1
-1	12	-4	15	-8	7	-8
1	12	20	31	38	37	36
$-\frac{1}{2}$	12	2	10	2	-2	0
	6	1	5	1	-1	
$\frac{1}{3}$	6	3	6	3	0	
	2	1	2	1		
$-\frac{1}{2}$	2	0	2	0		
	1	0	1			

$x = \{-\frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, \pm i\}$

(3) $2x^3 - 3x^2 - x + 12 = 0$

$\frac{1, 2, 3, 4, 6, 12}{1, 2}$

$-\frac{1}{2}, -\cancel{1}, \cancel{3}, -\cancel{3}, -\frac{3}{2},$

	2	−3	−1	12
−1	2	−5	4	8
1	2	−1	−2	10
$-\frac{3}{2}$	2	−6	8	0

$1 \quad -3 \quad 4$

$x = \frac{3 \pm \sqrt{9-16}}{2}$

$\frac{3 \pm i\sqrt{7}}{2}$

$x \in \left\{ -\frac{3}{2}, \frac{3 \pm i\sqrt{7}}{2} \right\}$

(4) $2x^3 - 7x^2 - 21x + 54 = 0$

$\frac{1, 2, 3, 6, 9, 18, 27, 54}{1, 2}$

$\frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}, \frac{-3}{1}, -\frac{3}{2}, \frac{+9}{2},$

	2	−7	−21	54
−1	2	−9	−12	66
1	2	−5	−26	28
$\frac{9}{2}$	2	2	−12	0

$1 \quad 1 \quad -6$

$(x+3)(x-2)$

$x \in \left\{ \frac{9}{2}, -3, 2 \right\}$

(5) $4x^4 - 12x^3 + x^2 + 14x - 3 = 0$

	4	−12	1	14	−3
−1	4	−16	17	−3	0
−1	4	−20	37	40	
1	4	−12	5	−8	
$\frac{3}{2}$	4	−10	2	0	

$x = \left\{-1, \frac{3}{2}, \frac{5 \pm \sqrt{17}}{4}\right\}$

(6) $2x^5 - 11x^4 + 4x^3 + 53x^2 - 48x - 36 = 0$

	2	−11	4	53	−48	−36
−1	2	−13	17	36	−84	48
1	2	−9	−5	48	0	−36
$-\frac{1}{2}$	2	−12	10	48	−72	0
3	2	−5	−11	20	12	0
3	2	1	−8	−4	0	
2	2	5	2	0		
−2	2	1	0			

$x = \left\{-\frac{1}{2}, \pm 2, 3, 3\right\}$

(9) $12x^4 - 56x^3 + 59x^2 + 9x - 18 = 0$

	12	−56	59	9	−18
-1	12	−68	127	−118	100
1	12	−44	15	24	6
$-\frac{1}{2}$	12	−62	90	−36	0
$\frac{2}{3}$	12	−48	27	27	0
$\frac{3}{2}$	12	38	2	12	0
3	12	−20	−1	6	0

$x \in \{3, \frac{3}{2}, \frac{2}{3}, -\frac{1}{2}\}$

(8) $3x^4 + 46x^3 - 30x^2 + 33x + 16 = 0$

	3	46	−30	33	16
−1	3	43	−73	106	−90
1	3	49	19	52	68
$-\frac{1}{3}$	3	45	−45	48	0
−16	3	−2	2	1	0

$x = \left\{-\frac{1}{3}, -16, \frac{1 \pm 3i}{2}\right\}$

(9) $16x^4 - 72x^2 + 81$

$(4x^2 - 9)(4x^2 + 9)$ $\quad x = \left\{\frac{3}{2}, -\frac{3}{2}, \pm\frac{3}{2}i\right\}$

$(2x - 3)(2x + 3)(4x^2 - 9)$

$x^2 = \frac{9}{4} \pm \frac{3}{2}$

(10) $10x^4 - 27x^3 - 2x^2 + 44x - 24 = 0$

$\frac{1, 2, 3, 4, 6, 8, 12, 24}{1, 2, 5, 10}$

	10	-27	-2	44	-24
-1	10	-37	35	9	-33
1	10	-17	-19	25	1
$\frac{6}{5}$	10	-15	-20	20	0
2	10	-7	-16	12	0

$x = \left\{ \frac{6}{5}, 2, \frac{-1 \pm \sqrt{17}}{4} \right\}$

(11) $6x^4 - 35x^3 + 64x^2 - 15x - 50 = 0$

	6	-35	64	-15	-50
-1	6	-41	105	-120	70
1	6	-29	35	20	-30
$-\frac{2}{3}$	6	-39	90	-75	0
$\frac{5}{2}$	6	-20	14	20	0

$x = \left\{ -\frac{2}{3}, \frac{5}{2}, 2 \pm i \right\}$

(12) $6x^4 + 13x^3 - 7x^2 - 26x - 10 = 0$ $\quad \frac{1, 2, 5, 10}{1, 2, 3, 6}$

	6	13	−7	−26	−10
-1	6	7	−14	−12	2
1	6	13	12	−14	−24
$-\frac{1}{2}$	6	10	−12	−20	0
	3	5	−6	−10	
$-\frac{5}{3}$	3	0	−6	0	

$x^2 - 2 = 0$ $\quad x \in \left\{-\frac{1}{2}, -\frac{5}{3}, \pm\sqrt{2}\right\}$

(13) $270x^2 - 453x + 287x - 40 = 0$

$270x^2 - 166x - 40 = 0$

$135x^2 - 83x - 20$

$\frac{83 \pm 133}{270}$ $\quad \frac{216}{270}$ $\quad \frac{50}{270} = \frac{5}{27}$

$\frac{10f}{135} = \frac{4}{25}$

$\div 27$ $\quad x \in \left\{\frac{4}{5}, \frac{5}{27}\right\}$

(14) $10x^4 - 130x^2 + 360 = 0$

$x^4 - 13x + 36 = 0$

$(x^2 - 4)(x^2 - 9)$

$x \in \{\pm 2, \pm 3\}$

(15) $24x^3 + 50x^2 - 29x - 10 = 0$

$\frac{1, 2, 5, 10}{1, 2, 3, 4, 6, 8, 12}$

	24	50	-29	-10
-1	24	26	-55	45
1	24	74	45	35
$\frac{2}{3}$	24	66	15	0
	8	22	5	

$x \in \{\frac{2}{3}, -\frac{5}{2}, -\frac{1}{4}\}$

$(2x + 5)(4x + 1)$

(16) $192x^4 + 40x^3 - 66x^2 + 9x = 0$

	192	40	-66	9
-1	192	-152	86	77
1	192	232	166	175
$-\frac{3}{4}$	192	-104	12	0

$x \in \{0, -\frac{3}{4}, \frac{1}{4}, \frac{3}{8}\}$

(17) $75x^3 - 10x^2 - 96x - 32 = 0$

	75	−10	−96	−32
−1	75	−85	−11	−21
1	75	65	−31	−63
$-\frac{2}{5}$	75	−40	−80	0

$75x^2 - 40x - 80$

$15x^2 - 8x - 16 = 0$

$(3x - 4)(5x + 4)$

$x \in \{-\frac{2}{5}, \frac{4}{3}, -\frac{4}{5}\}$

(18) $12x^3 + 112x^2 + 343x + 343 = 0$

	12	112	343	343
−1	12	100	243	100
1	12	124	567	910
$-\frac{7}{3}$	12	84	147	0

$12x^2 + 84x + 147$

$4x^2 + 28x + 49$

$(2x + 7)(2x + 7)$

$x \in \{-\frac{7}{3}, -\frac{7}{2}, -\frac{7}{2}\}$

3·5·5

(19) $128x^3 + 48x^2 - 200x - 75 = 0$

$\frac{1, 3, 5, 15, 25, 75}{1, 2, 4, 8, 16, 32, 64, 128}$

	128	48	−200	−75
−1	128	−80	−120	45
1	128	176	−24	−99
$-\frac{3}{8}$	128	0	−200	0
	16	0	−25	

$-\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{8}, \frac{3}{2}, -\frac{3}{2}$

$-\frac{3}{4}, -\frac{3}{8}, -\frac{5}{2}, -\frac{5}{8}, \frac{5}{4}, -\frac{5}{4}$

$16x^2 - 25 = 0$

$x = \pm\frac{5}{4}$

$x = \left\{-\frac{3}{8}, \pm\frac{5}{4}\right\}$

(20) $36x^4 - 24x^3 - 95x^2 + 75x = 0$

$\frac{1, 3, 5, 15, 25, 75}{1, 2, 3, 4, 6, 9, 12, 36}$

	36	−24	−95	75
−1	36	−60	−35	110
1	36	12	−83	−8
$\frac{3}{2}$	36	30	−50	0
	18	15	−25	

$18x^2 + 15x - 25$

$(3x + 5)(6x - 5)$

$x = \left\{0, \frac{3}{2}, -\frac{5}{3}, \frac{5}{6}\right\}$